$$\sum_{n=0}^{\infty} n^2[J_n'(ne)]^2=\frac{1}{4(1-e^2)^{5/2}}\left(1+\frac{3e^2}{4}\right),$$

$$\sum_{n=0}^{\infty} n^4 J_n^2(ne)=\frac{e^2}{4(1-e^2)^{13/2}}\left(1+\frac{37}{4}e^2+\frac{59}{8}e^4+\frac{27}{64}e^6\right).$$

Substitution of (A3) into the sum of the reduced series (A1) yields

$$\sum_{n=1}^{\infty} g(n,e)=\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}},$$

which is the same as $f(e)$ as calculated in (17).

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Quasiparticles and the Born Series*

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Perturbation theory always works in nonrelativistic scattering theory, unless composite particles are present. By "composite particle" is meant a bound state or resonance, or one that would exist for an interaction of opposite sign; in fact, this provides a precise definition of resonances. It follows that if fictitious elementary particles (quasiparticles) are first introduced to take the place of all composite particles, then perturbation theory can always be used. There are several ways of accomplishing this, one of which corresponds to the N/D method. In order to prove these results it is necessary to make a detailed study of the eigenvalues of the scattering kernel, and as a by-product we obtain new proofs of the applicability of the Fredholm theorems to scattering theory, of the convergence of the Born series at high energy, of the Bargmann-Schwinger theorem on the number of bound states, of the Pais-Jost theorem on the identity of the Jost function with the Fredholm determinant, and of Levinson's theorem. We also give explicit formulas for binding energies and phase shifts in potential theory, using first-order perturbation theory after insertion of a single quasiparticle; these formulas work well for the lowest bound state and the S-wave scattering length of the Yukawa potential, and give precisely 13.6 eV for the hydrogen atom binding energy.

I. INTRODUCTION

THIS is the second of a series of papers, in which we hope to develop a practicable method of calculating strong interaction processes.

In our first paper[1] it was proven that any given nonrelativistic Hamiltonian H can be rewritten to introduce fictitious elementary particles (quasiparticles) which did not appear in H. The new Hamiltonian $\mathbf{H}$ yields precisely the same physical predictions as H, provided that when we put the quasiparticles into the unperturbed part, we also modify the interaction term according to certain rules. These matters are reviewed in Sec. II.

We also remarked in A that such quasiparticles can be introduced very freely, without any reference to physically real particles, and also without any point. But their introduction can be the crucial step in practical calculations, for such calculations can always be done by perturbation methods unless composite particles are present. If we introduce a quasiparticle corresponding to each composite particle, then we get a new (but physically equivalent) theory in which there are no composites, but only real and fictitious elementary particles, so that perturbation theory works. What actually happens is that the modification of the Hamiltonian forced upon us by the introduction of a quasiparticle weakens the original interaction enough to remove the divergence of the Born series associated with the corresponding composite particle.[2] Seen in this way, the strength of a given coupling should never make us despair of applying perturbation theory; a very strong interaction merely gives rise to many composite particles, and, hence, forces us to introduce a large number of quasiparticles before we start using the Born series.

I believe that this approach will make perturbation theory universally applicable, even to the full relativistic series of Feynman diagrams.[3] The purpose of this paper is to demonstrate that this conjecture is, indeed, correct within the limited proving ground of nonrelativistic two-body scattering theory.

It is shown in Sec. III that the Born series will diverge if and only if there are composite particles present, and

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[1] S. Weinberg, Phys. Rev. **130**, 776 (1963); this article will be referred to as A.
[2] A more general approach to the problem of obtaining a convergent perturbation series has been suggested by M. Rotenberg (to be published). Our approach seems to correspond to his if the operator he calls "$J-1$" is chosen to be separable; otherwise the quasiparticle interpretation is inapplicable.
[3] Some preliminary steps in this direction are reported by S. Weinberg, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 683.

in Secs. IV and V (or XI) that the divergence can be cured by introducing quasiparticles corresponding to each composite particle. But we have to be careful with what we mean by this. What matters for convergence at a given energy W is, of course, not just a bound state that might happen to be precisely at W, but rather it is the whole denumerably infinite set of energy eigenstates $|\Psi_\nu\rangle$ which can be *shifted* to W by dividing the interaction by a real or complex number $\eta_\nu(W)$. The physical Born series diverges at energy W if and only if some $\eta_\nu(W)$ lies outside the unit circle, and all such divergences can be cured by introducing quasiparticles in correspondence with each $\eta_\nu(W)$ that ever leaves the unit circle as W increases from $-\infty$ to $+\infty$. At the particular energy where some $\eta_\nu(W)$ leaves the unit circle we have either an actual bound state or resonance, or one that would be present for an interaction of opposite sign; a bound state or a resonance, thus, always signifies the beginning of a divergence of the Born series. In fact, we may take that this is a new definition of a resonance, which agrees with all previous definitions for narrow resonances, and which continues to be precise and significant even for broad ones.

The interpretation and behavior of the $\eta_\nu(W)$ are discussed at length in Sec. VI, and examples are presented in Sec. VII. The most important result obtained there is that only a finite number of $\eta_\nu(W)$ ever leave the unit circle, so that at most a finite number of quasiparticles need be introduced to make perturbation theory work at all energies. (The energy $W=0$ must be excluded for long-range forces.) Also, as by-products of our study of the $\eta_\nu(W)$, we are able to offer new proofs of the applicability of the Fredholm theorems to scattering theory (Sec. III), of the convergence of perturbation theory for large $|W|$ (Sec. VI), of Levinson's theorem (Sec. VIII), of the Bargmann-Schwinger theorem on the number of bound states (Sec. IX), and of the Pais-Jost theorem on the identity of the Jost function with the Fredholm determinant (Sec. XI). We also give an upper bound on the binding energy of any state bound by any short-range potential. (Sec. IX.)

In order to show that perturbation theory can really be made to work, we have derived in Sec. X general formulas for the phase shifts and lowest bound states given in potential theory by using first-order perturbation theory after insertion of a single crude quasiparticle. The scattering length and binding energy have then been evaluated for the S-wave Yukawa case, with gratifying results. More extensive nonrelativistic calculations are now under way.

The interested reader with little taste for details is advised to read Sec. II, skim III and IV, read V, and skim VI, VII, and X. Section II should make it unnecessary to refer back to A. Our next paper will use the quasiparticle method to solve the multibody problem, and the following one will extend the method to relativistic particle physics.

II. QUASIPARTICLES

We begin by reviewing the parts of A which form our present starting point.

Consider a Hamiltonian H, which is split into an unperturbed part H_0 and an interaction V. We shall not be too specific about the system considered, although some methods used in this paper need revision for more than two particles. It will be assumed that H_0 has only continuum eigenstates, which are characterized by the energy $E>0$, and perhaps also by other variables (angular momenta, isospins, etc.) labeled by an index n:

$$H_0|En\rangle=E|En\rangle; \tag{1}$$

$$\langle E'n'|En\rangle=\delta_{n'n}\delta(E'-E). \tag{2}$$

We treat n as if it were discrete, though nothing is changed if n also refers to angles, etc.
All observables in such a theory can be obtained from an operator $T(W)$, defined for all complex W by

$$T(W)=V+T(W)[W-H_0]^{-1}V = V+V[W-H_0]^{-1}T(W). \tag{3}$$

In particular, the S matrix is

$$S_{n'n}(E)=\delta_{n'n}-2\pi i\langle En'|T(E+i\epsilon)|En\rangle, \tag{4}$$

where $E>0$ and $\epsilon\to+0$; the bound-state energies are at the poles of $T(W)$.

If the kernel $[W-H_0]^{-1}V$ of Eq. (3) is sufficiently small at some energy W, then $T(W)$ can be expressed as the Born series:

$$T(W)=V+V[W-H_0]^{-1}V+\cdots. \tag{5}$$

But no term in this series has any poles in W. Hence, the series must diverge for W near bound state energies. We shall see that it also diverges near resonances, and, in a certain sense, nowhere else.

In order to cure this divergence of the Born series, we consider instead of (3) a new integral equation

$$T_1(W)=V_1(W)+T_1(W)[W-H_0]^{-1}V_1(W), \tag{6}$$

where $V_1(W)$ is a "reduced" interaction

$$V_1(W)=V-V|\Gamma(W)\rangle\langle\bar{\Gamma}(W)|V, \tag{7}$$

and $|\Gamma(W)\rangle$ and $\langle\bar{\Gamma}(W)|$ are *any* linear combinations of the continuum states $|En\rangle$. It is shown in A that $T(W)$ can be expressed in terms of $T_1(W)$ by

$$T(W)=T_1(W)+N^{-2}(W)T_1(W)|\Gamma(W)\rangle \times\Delta(W)\langle\bar{\Gamma}(W)|T_1(W), \tag{8}$$

$$\Delta(W)=[1-J(W)]^{-1}, \tag{9}$$

$$J(W)\equiv N^{-2}(W)\langle\bar{\Gamma}(W)|V_1(W) \times[W-H_0]^{-1}T_1(W)|\Gamma(W)\rangle, \tag{10}$$

$$N(W)\equiv 1-\langle\bar{\Gamma}(W)|V|\Gamma(W)\rangle. \tag{11}$$

The factors $N(W)$ always cancel in practical calculations.

It was also observed in A that a formula similar to (8) would arise in a theory in which the actual interaction were V_1, but in which H_0 had a discrete "elementary particle" eigenstate. The operator $T_1(W)$ corresponds to the sum of "proper" graphs excluding one-elementary-particle exchange (in the s channel); $\langle\bar{\Gamma}|T_1$ corresponds to the complete proper incoming vertex; Δ to the complete unrenormalized elementary particle propagator; and $T_1|\Gamma\rangle$ to the complete proper outgoing vertex. Equation (8) has the obvious significance that any graph either belongs in T_1, or it arises from one-elementary-particle exchange. Except for an over-all normalization factor, formula (9) for the propagator $\Delta(W)$ is the same as would arise in a theory in which there actually was an elementary particle (with infinite bare mass). Hence, we refer to the steps leading to (8) and (9) as the introduction of a quasiparticle into the theory.[4]

The point of introducing the quasiparticle is that the new potential V_1 may be sufficiently weak for the convergence of the new Born series

$$T_1(W)=V_1(W)+V_1(W)[W-H_0]^{-1}V_1(W)+\cdots, \quad (12)$$

which can be used with Eqs. (8)–(10) to calculate $T(W)$. The bound state pole in $T(W)$ then must arise in the propagator $\Delta(W)$. To make (12) converge, it is clear that we must try to adjust $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$ so that the original interaction V is well approximated by the separable interaction $V|\Gamma\rangle\langle\bar{\Gamma}|V$.

It is also shown in A that the Fredholm determinants corresponding to (3) and (6) are related by

$$D(W)=D_1(W)\Delta^{-1}(W). \quad (13)$$

The quasiparticle (or "Schmidt") method, therefore, separates out just one troublesome factor from the Fredholm determinant.

III. CONVERGENCE OF THE BORN SERIES

Our task in this paper is to show that the introduction of quasiparticles in close correspondence with real bound states or resonances can always succeed in making the Born series converge. As our starting point, we shall prove in this section that the Born series (5) or (12) will converge if and only if the kernel has no eigenvalue outside the unit circle. This is essentially the criterion already given by Jost and Pais[5] for the special case of a local short-range potential; in this section we will extend this theorem to almost any interesting two-body interaction.

We define the set of eigenvalues $\eta_\nu(W)$ and eigenstates $|\psi_\nu(W)\rangle$ by the equation

$$[W-H_0]^{-1}V|\psi_\nu(W)\rangle=\eta_\nu(W)|\psi_\nu(W)\rangle, \quad (14)$$

with the understanding that $|\psi_\nu(W)\rangle$ must actually be in the Hilbert space, i.e., have finite norm. For the time being, we keep W negative or complex, although W will later be allowed to approach the positive real axis. We shall prove that the series (5) for $T(W)$ converges at energy W if and only if the eigenvalues satisfy

$$|\eta_\nu(W)|<1, \quad (\text{all } \nu). \quad (15)$$

The same theorem applies to the reduced interaction $V_1(W)$. The convergence is "relatively uniform,"[6] a term explained below. Properties of the $\eta_\nu(W)$, and their composite-particle interpretation, are discussed in Sec. VI.

The necessity of condition (15) for any sort of convergence is obvious, because the series (5), when applied to any eigenstate, gives

$$T(W)|\psi_\nu(W)\rangle=\sum_{m=0}^{\infty}\eta_\nu{}^m(W)V|\psi_\nu(W)\rangle. \quad (16)$$

This clearly diverges if $|\eta_\nu(W)|\geqq 1$.

In order to prove that condition (15) is sufficient as well as necessary for convergence, we need to use the Fredholm theorems to study the behavior of the T operator for interaction λV:

$$T(W,\lambda)=\lambda V+\lambda V[W-H_0]^{-1}T(W,\lambda), \quad (17)$$

as a function of the complex coupling parameter λ. The most convenient condition I know which would allow some form of Fredholm theory to be applied to a given kernel K is that K be "L^2," i.e., that $\mathrm{Tr}KK^\dagger$ be finite.[7] In our case, this condition becomes $\tau(W)<\infty$, where

$$\tau(W)\equiv\mathrm{Tr}\left\{\frac{1}{|W-H_0|^2}V^2\right\}=\sum_n\int_0^\infty dE\frac{\langle En|V^2|En\rangle}{|W-E|^2}. \quad (18)$$

We will show in Appendix A that $\tau(W)$ is actually finite[8]

[4] A similar approach is followed by M. T. Vaughan, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961). A somewhat different way of making composite particles seem elementary is being developed by A. Salam, Nuovo Cimento **25**, 224 (1962), and to be published. See also J. C. Howard and B. Jouvet, *ibid.* **18**, 466 (1960), and R. Acharya, *ibid.* **24**, 870 (1962). For other interesting comments on the relation of bound to elementary particles, see F. E. Low, *ibid.* **25**, 678 (1962) and E. G. P. Rowe (to be published). These authors are all primarily concerned with questions of principle about the meaning of elementarity, rather than with the point that seems to me to be most vital, that the representation of composite particles as if they were elementary can make perturbation theory work.

[5] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951). See also W. Kohn, Rev. Mod. Phys. **26**, 292 (1954).

[6] E. H. Moore, New Haven Mathematical Colloquium (1910) (unpublished), pp. 1–150.

[7] All results needed for such kernels may be found in standard works, such as *Integral Equations*, by F. Smithies (Cambridge University Press, New York, 1958); see particularly Secs. 2.5, 2.6, and Chap. VI. Other names sometimes given to such kernels include "Hilbert-Schmidt kernel," "Fredholm operator," etc.

[8] Originally I thought $\tau(W)$ was infinite, and the preprint of this paper was based instead on the assumption $\tau_2(W)<\infty$, where $\tau_2(W)$ [previously called $\tau(W)$] is defined by Eq. (B1) in Appendix B. I am very grateful to Professor N. Kroll for bringing this mistake to my attention, because considerable simplifications were thus made possible. It is conceivable that for some V, $\tau_2(W)$

for the full three-dimensional scattering problem with a local potential $V(\mathbf{r})$ if and only if

$$\int |V(\mathbf{r})|^2 d^3\mathbf{r} < \infty . \tag{19}$$

If we restrict ourselves to a single partial wave, the trace is finite if and only if

$$\int_0 |V(r)|^2 r^2 dr < \infty ; \quad \int^\infty |V(r)|^2 dr < \infty . \tag{20}$$

These conditions for validity of the Fredholm theory may be contrasted with those given by Jost and Pais[9] for the three-dimensional case,

$$\int_0^\infty |V(r)| r dr < \infty ; \quad r^2 V(r) \text{ bounded}, \tag{21}$$

or by Newton[10] for partial waves

$$\int_0 |V(r)| r dr < \infty ; \quad \int^\infty |V(r)| dr < \infty . \tag{22}$$

Our conditions are very much weaker. In particular, (20) holds *even for the Coulomb potential*, where (22) certainly does not. More important, condition (18) is applicable even to a nonlocal interaction; only the high-energy behavior of $\langle En|V^2|En\rangle$ is relevant in deciding whether the Fredholm theorems hold. Presumably the reason previous authors had to impose such restrictive conditions, and then still had to go through all the trouble of reproving the Fredholm theorems, was that they insisted on taking W real and positive throughout, whereas we stay off the positive real axis until the end.

[But a cloud on the horizon should keep us from expecting too much from the Fredholm method; the trace $\tau(W)$ turns out to be infinite as soon as we turn to three-body processes. I shall show in the next paper of this series that the Fredholm method does, in fact, break down in the multiparticle case, and that the quasiparticle method survives untarnished.]

Assuming that $[W-H_0]^{-1}V$ is an L^2 kernel, we may now apply standard theorems[7] to find the properties of its resolvent, which is just $\lambda^{-1}[W-H_0]^{-1}T(W,\lambda)$. In this way we learn that

(i) Each matrix element $\langle En|T(W,\lambda)|E'n'\rangle$ is a meromorphic function of λ, with poles at the "characteristic" values, $\lambda=\eta_\nu^{-1}(W)$, the $\eta_\nu(W)$ being defined by (14).

(ii) At least for $|\lambda|^2\tau(W)<1$, the matrix elements of $T(W,\lambda)$ are given by the (absolutely) convergent series

$$\langle En|T(W,\lambda)|E'n'\rangle = \sum_{m=1}^\infty \lambda^m \langle En|V\{[W-H_0]^{-1}V\}^{m-1}|E'n'\rangle. \tag{23}$$

(iii) For $\lambda \neq \eta_\nu^{-1}(W)$, the resolvent is itself an L^2 kernel, i.e., $\sigma(W,\lambda)<\infty$, where

$$\sigma^2(W,\lambda) \equiv \sum_{nn'} \int_0^\infty dE \int_0^\infty dE' \times |\langle En|T(W,\lambda)|E'n'\rangle|^2 / |W-E|^2. \tag{24}$$

The proof of our theorem is now almost obvious; we need only recall the classic theorem that a function of λ analytic within some circle $|\lambda|<\lambda_1$ is given in that circle by its Taylor series expansion. From (i) we see that the radius of the circle of convergence is just $\lambda_1=|\eta_1^{-1}(W)|$, where $\eta_1(W)$ is the eigenvalue of greatest modulus. And from (ii) we see that the Taylor series for $T(W,\lambda)$ is just the Born series (23).

We are also interested in the uniformity of the convergence, so it will pay us to go through this argument in detail,[11] and to actually set an upper bound on matrix elements of the remainder of the Born series,

$$R^{(P)}(W,\lambda) \equiv T(W,\lambda) - \sum_{m=1}^P \lambda^m V\{[W-H_0]^{-1}V\}^{m-1}. \tag{25}$$

Suppose that $|\lambda|<|\eta_1^{-1}(W)|$, and let C be a circle $|z|=\rho$, with radius ρ chosen so that

$$|\lambda|<\rho<|\eta_1^{-1}(W)|.$$

The matrix elements $\langle En|T(W,z)|E'n'\rangle$ are analytic inside and on C, so Cauchy's theorem gives

$$\frac{\lambda^{P+1}}{2\pi i}\oint_C \frac{\langle En|T(W,z)|E'n'\rangle}{z^{P+1}(z-\lambda)}dz = \langle En|R^{(P)}(W,\lambda)|E'n'\rangle. \tag{26}$$

[The term arising from the pole at $z=0$ can be determined by inspection of (23) to be the matrix element of

might be finite and $\tau(W)$ infinite, because Appendix B shows that $\tau_2(W)<\tau(W)$. In this event, all important results could be proven as in the preprint by performing a nonunitary similarity transformation of $[W-H_0]^{-1}V$ into the kernel $K(W)$ defined by Eq. (78), by using reference 7 for $K(W)$, and then transforming back. I have just received a preprint by L. Brown, D. I. Fivel, B. W. Lee, and R. J. Sawyer, in which precisely this is done in potential theory.

[9] See reference 5; see also N. N. Khuri, Phys. Rev. **107**, 1148 (1957).

[10] R. Newton, J. Math. Phys. **1**, 319 (1960). The effort expended by Newton and by Jost and Pais in proving the convergence of the Fredholm numerator and denominator is made unnecessary by use of the theorems described in reference 7.

[11] We follow the treatment of E. J. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1950), 4th ed., p. 93.

the sum in (25).] This has the upper bound

$$|\langle En|R^{(P)}(W,\lambda)|E'n'\rangle| \leqq \epsilon_P \int_0^{2\pi} |\langle En|T(W,\rho e^{i\theta})|E'n'\rangle| d\theta; \quad (27)$$

$$\epsilon_P \equiv \frac{\rho}{2\pi(\rho-|\lambda|)}\left|\frac{\lambda}{\rho}\right|^{P+1},$$

which obviously vanishes as $P\to\infty$. So (23) holds within the circle of convergence $|\lambda|<|\eta_1^{-1}(W)|$. Condition (15) ensures that the actual coupling parameter $\lambda=1$ lies within this circle of convergence, so (15) is a sufficient condition for convergence of any fixed element of the Born series.

We can now also use (27) to determine the uniformity properties of the convergence. From (iii) and the Schwarz inequality, we see that

$$\sum_{nn'}\int_0^\infty dE\int_0^\infty dE' |\langle En|R^{(P)}(W,\lambda)|E'n'\rangle|^2/|W-E|^2 \leqq \epsilon_P^2 M^2(W,\rho); \quad (28)$$

$$M(W,\rho)\equiv\int_0^{2\pi}\sigma(W,\rho e^{i\theta})d\theta.$$

In other words, $[W-H_0]^{-1}R^{(P)}(W,\lambda)$ is itself ϵ_P times an L^2 kernel independent of P. Such convergence is called "relatively uniform."[6] The most important consequence is that for any pair of wave packets $|\psi_a\rangle$ and $|\psi_b\rangle$ normalized according to

$$\langle\psi_a|\psi_a\rangle=\langle\psi_b|\psi_b\rangle=1, \quad (29)$$

the matrix element of the remainder $R^{(P)}$ has an upper bound

$$|\langle\psi_a|[W-H_0]^{-1}R^{(P)}(W,\lambda)|\psi_b\rangle| \leqq \epsilon_P M(W,\rho), \quad (30)$$

which vanishes as $P\to\infty$, and which is *independent* of $|\psi_a\rangle$ and $|\psi_b\rangle$.

We must now show that these results can be extended to scattering problems, where $W=E+i\epsilon$, with $E>0$ and $\epsilon\to+0$. At first we might anticipate trouble here, because $\tau(E+i\epsilon)$, $\sigma(E+i\epsilon,\lambda)$, and $M(E+i\epsilon,\rho)$ all become infinite for $\epsilon=0$. But this is misleading. We have already shown that the true radius of convergence at $E+i\epsilon$ is not $\tau^{-1/2}(E+i\epsilon)$, but the generally larger value $|\eta_1(E+i\epsilon)|^{-1}$, and this approaches a finite limit almost everywhere as $\epsilon\to 0$. (The only exception arises at $E=0$ for long-range forces; see Sec. VI.) To see that this limit is the radius of convergence of the limit of $T(E+i\epsilon,\lambda)$, we need only set $W=E+i\epsilon$ in (27) and let $\epsilon\to 0$. All arguments go through as before, provided that fixed matrix elements of $T(E+i\epsilon,\lambda)$ for $\lambda\leqq\rho<|\eta_1^{-1}(E+i\epsilon)|$ approach finite limits as $\epsilon\to 0$. Whether or not this is the case lies outside the scope of our work, but on physical grounds we should expect $T(E+i\epsilon,\lambda)$ to stay finite as $\epsilon\to 0+$, unless there were an energy eigenstate at E for the coupling constant λ, i.e., unless $\lambda\eta_\nu(E+i\epsilon)=1$ for some ν; and this is impossible, because $\lambda<|\eta_1^{-1}|$. (Similar arguments can be used to justify all of the Fredholm results for $\epsilon\to 0$.)

To summarize: Condition (15) is necessary and sufficient for convergence of fixed matrix elements of the Born series at any fixed W. The convergence is uniform [in the sense of (29) and (30)] with respect to the state vectors defining the matrix element. It is uniform in W in any closed region within which (15) holds and $T(W,\lambda)$ exists. Our basic assumption is the existence of $\tau(W)$, defined by (18).

All these results apply equally to the reduced Born series (12), if we substitute $V_1(W)$ everywhere for V. The trace $\tau(W)$ must be replaced by

$$\mathrm{Tr}\left\{\frac{1}{|W-H_0|^2}V_1(W)V_1^\dagger(W)\right\}$$
$$=\tau(W)-2\,\mathrm{Re}\langle\bar\Gamma(W)|V^2\frac{1}{|W-H_0|^2}V|\Gamma(W)\rangle$$
$$+\langle\Gamma(W)|V\frac{1}{|W-H_0|^2}V|\Gamma(W)\rangle\langle\bar\Gamma(W)|V^2|\Gamma(W)\rangle.$$

If $\tau(W)$ exists, then usually so will this; at any rate this point can always be checked easily by direct calculation.

IV. EFFECT OF THE QUASIPARTICLE

We now know that a divergence of the Born series occurs when the kernel $[W-H_0]^{-1}V$ has an eigenvalue outside the unit circle, and that it will be cured by the introduction of a quasiparticle if the resulting reduced kernel $[W-H_0]V_1(W)$ does not have any eigenvalues outside the unit circle. So we are naturally led to study the spectrum of the eigenvalues χ of the reduced kernel, defined by

$$[W-H_0]^{-1}V_1(W)|\Phi\rangle=\chi|\Phi\rangle. \quad (32)$$

We will now show that these eigenvalues χ form two classes:

(A) All roots of the equation

$$\Delta(W,\chi^{-1})=0, \quad (33)$$

where

$$\Delta(W,\lambda)\equiv 1+\langle\bar\Gamma(W)|V[W-H_0]^{-1}T(W,\lambda)|\Gamma(W)\rangle. \quad (34)$$

[If the interaction were λV, the T operator would be $T(W,\lambda)$, and we shall see that the propagator would be $\Delta(W,\lambda)$.]

(B) All $\eta_\nu(W)$ for which

$$\langle\bar\Gamma(W)|V|\Psi_\nu(W)\rangle=0, \quad (35)$$

or

$$\langle\Psi_\nu(W^*)|V|\Gamma(W)\rangle=0. \quad (36)$$

To solve (32), we note that it can be written as an

inhomogeneous integral equation

$$\chi|\Phi\rangle=C(W)[W-H_0]^{-1}V|\Gamma(W)\rangle + [W-H_0]^{-1}V|\Phi\rangle, \quad (37)$$

where

$$C(W)=-\langle\bar{\Gamma}(W)|V|\Phi\rangle. \quad (38)$$

There are then just three familiar possibilities:

(A) The eigenvalue χ is not one of the $\eta_\nu(W)$. Then Eq. (17) shows that (37) has the unique solution,

$$|\Phi\rangle=C(W)[W-H_0]^{-1}T(W,\chi^{-1})|\Gamma(W)\rangle. \quad (39)$$

This satisfies Eq. (32) if (38) holds. Substituting (39) in (38), we see that if $C(W)$ and $|\Phi\rangle$ are not to vanish then χ must satisfy (33).

(B1) The eigenvalue χ is equal to some $\eta_\nu(W)$, and $C(W)=0$. Then (37) shows that $|\Phi\rangle$ must be just

$$|\Phi\rangle=|\Psi_\nu(W)\rangle. \quad (40)$$

This solves the eigenvalues problem if (38) vanishes, i.e., if (35) holds.

(B2) The eigenvalue χ is equal to some $\eta_\nu(W)$, but $C(W)\neq 0$. Then it is well known[12] that (37) has a solution if and only if the inhomogeneous term $[W-H_0]^{-1}V|\Gamma(W)\rangle$ is orthogonal to the eigenvector of $\{[W-H_0]^{-1}V\}^\dagger$ with eigenvalue $\eta_\nu^*(W)$, which is just $V|\Psi_\nu(W^*)\rangle$; hence, (37) has a solution if

$$0=\langle\Psi_\nu(W^*)|V[W-H_0]^{-1}V|\Gamma(W)\rangle =\eta_\nu(W)\langle\Psi_\nu(W^*)|V|\Gamma(W)\rangle,$$

so (36) is sufficient for a solution to exist. That (36) is necessary follows by taking the matrix element of (37) with $\langle\Psi_\nu(W^*)|V$. If $C(W)|\Phi_0\rangle$ is some particular solution of (37), then the general solution is

$$|\Phi\rangle=C(W)[|\Phi_0\rangle-\sum_\mu{}'\alpha_\mu|\Psi_\mu(W)\rangle], \quad (41)$$

where the sum is restricted to μ such that $\eta_\mu(W)=\eta_\nu(W)$. Equation (38) can always be satisfied by choosing that α_μ as solutions of

$$\sum_\mu{}'\langle\bar{\Gamma}(W)|V|\Psi_\mu(W)\rangle\alpha_\mu=\langle\bar{\Gamma}(W)|V|\Phi_0\rangle+1$$

unless all $\langle\bar{\Gamma}(W)|V|\Psi_\mu(W)\rangle$ vanish, in which case we are back in case B1. Q.E.D.

We also promised to show that $\Delta(W,\lambda)$ is the propagator for an interaction λV. To see that this is true for $\lambda=1$, we note from (8) that

$$\langle\bar{\Gamma}(W)|T(W)|\Gamma(W)\rangle=\langle\bar{\Gamma}(W)|T_1(W)|\Gamma(W)\rangle +N^{-2}(W)\langle\bar{\Gamma}(W)|T_1(W)|\Gamma(W)\rangle^2\Delta(W).$$

Using (9) and (10) we can eliminate the $\langle\bar{\Gamma}|T_1|\Gamma\rangle$ and N, and obtain

$$\Delta(W)=1+\langle\bar{\Gamma}(W)|V[W-H_0]^{-1}T(W)|\Gamma(W)\rangle. \quad (42)$$

If the interaction were λV, and the reduced interaction were λV_1, then the propagator could be derived from (42) by substituting

$$V\to\lambda V,\quad T(W)\to T(W,\lambda),$$
$$|\Gamma\rangle\to\lambda^{-1/2}|\Gamma\rangle,\quad \langle\bar{\Gamma}|\to\lambda^{-1/2}\langle\bar{\Gamma}|.$$

This then gives (34).

We could have anticipated our solution of this eigenvalue problem. For the operator $T_1(W,\chi^{-1})$ obviously has a pole when χ is one of the eigenvalues of the reduced kernel. But then we see from (8) that either the propagator $\Delta(W,\chi^{-1})$ must then vanish, or $T(W,\chi^{-1})$ will also have the same pole, in which case χ would have to be one of the $\eta_\nu(W)$.

Formula (34) for $\Delta(W,\lambda)$ becomes particularly simple if the vertices happen to be chosen as linear combinations of a set of nondegenerate eigenvectors Ψ_ν:

$$|\Gamma(W)\rangle=\sum_\nu g_\nu(W)|\Psi_\nu(W)\rangle, \quad (43)$$

$$\langle\bar{\Gamma}(W)|=\sum_\nu\bar{g}_\nu(W)\langle\Psi_\nu(W^*)|. \quad (44)$$

For then we can use the orthogonality relation

$$\langle\Psi_\nu(W^*)|V|\Psi_\mu(W)\rangle=\delta_{\nu\mu}h_\mu(W), \quad (45)$$

since (14) and its adjoint [with Eq. (87)] show that

$$\eta_\mu(W)\langle\Psi_\nu(W^*)|V|\Psi_\mu(W)\rangle =\langle\Psi_\nu(W^*)|V(W-H_0)^{-1}V|\Psi_\mu(W)\rangle =\eta_\nu(W)\langle\Psi_\nu(W^*)|V|\Psi_\mu(W)\rangle.$$

Applying (43)–(45) to (34) we see that

$$\Delta(W,\lambda)=1+\lambda\sum_\nu\frac{G_\nu(W)\eta_\nu(W)}{1-\lambda\eta_\nu(W)}, \quad (46)$$

where

$$G_\nu(W)=g_\nu(W)\bar{g}_\nu(W)h_\nu(W) =\frac{\langle\Psi_\nu(W^*)|V|\Gamma(W)\rangle\langle\bar{\Gamma}(W)|V|\Psi_\nu(W)\rangle}{\langle\Psi_\nu(W^*)|V|\Psi_\nu(W)\rangle} =1-\frac{\langle\Psi_\nu(W^*)|V_1(W)|\Psi_\nu(W)\rangle}{\langle\Psi_\nu(W^*)|V|\Psi_\nu(W)\rangle}. \quad (47)$$

[A typical function $\Delta(W,\chi^{-1})$ is plotted for real χ in Fig. 1.] Our previous result may now be stated: The eigenvalues of the reduced kernel are the roots of $\Delta(W,\chi^{-1})=0$, plus any $\eta_\nu(W)$ for which $g_\nu(W)$or $\bar{g}_\nu(W)$ vanishes, or, in other words, for which

$$G_\nu(W)=0. \quad (48)$$

Hence, the spectrum of the reduced kernel depends solely upon the coefficients $G_\nu(W)$. If there are p nonzero $G_\nu(W)$, then the reduced eigenvalues are all $\eta_\nu(W)$ except the p for which $G_\nu(W)\neq 0$, plus the p roots of Eq. (33); hence, no eigenvalues are lost or gained. A choice (43), (44) of bare vertices is acceptable if all roots of (33) lie within the unit circle, and if all η_ν for which $G_\nu=0$ are already inside the unit circle. Two possible such choices are offered in Secs. V and XI.

[12] See, for example, reference 7, p. 50.

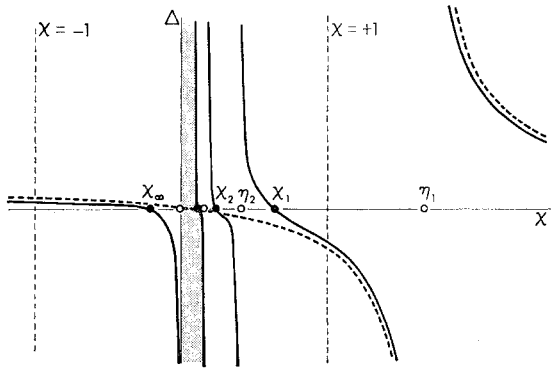

FIG. 1. The propagator $\Delta(W,\chi^{-1})$ as a function of χ, for some typical energy $W<0$. The dashed line corresponds to the "ideal" choice of Sec. V, and the solid lines correspond to some choice which is nearly ideal, and which does succeed in rendering the Born series convergent. The open dots indicate the original eigenvalues $\eta_\nu(W)$, while the black dots denote the eigenvalues $\chi_\nu(W)$ of the reduced kernel. The original Born series (5) diverged here because $\eta_1>1$, but the reduced series (12) converges because $|\chi_\nu|<1$ for all ν. The point $\chi=0$ is shown as an open dot because it is a point of accumulation of the η_ν; for this reason it is too difficult to represent what happens in the shaded region. We have drawn this diagram for a typical attractive interaction. A repulsive interaction would have all η_ν negative, while an interaction neither purely attractive nor purely repulsive would have η_ν on both sides of $\eta=0$.

Before closing this section we must note that the actual propagator is

$$\Delta(W)=1+\sum_\nu G_\nu(W)\frac{\eta_\nu(W)}{1-\eta_\nu(W)}. \tag{49}$$

We can see from (49) or (42) that $\Delta(W)$ has poles at the bound-state energies $-B$, which are determined by the condition that for some ν

$$\eta_\nu(-B)=1. \tag{50}$$

Together with Eq. (14), this says that $|\Psi_\nu(-B)\rangle$ satisfies the usual Schrödinger equation,

$$[H_0+V]|\Psi_\nu(-B)\rangle=-B|\Psi_\nu(-B)\rangle.$$

It is also possible in scattering problems for $\eta_\nu(W)$ to pass through the unit circle, in which case it will be complex, and $|\Psi_\nu(E+i\epsilon)\rangle$ will represent a resonance. These matters are discussed fully in Secs. VI and VII.

V. THE CHOICE OF BARE VERTICES

The "ideal" choice of bare vertices is

$$|\Gamma(W)\rangle\propto|\Psi_1(W)\rangle, \tag{51}$$

$$\langle\bar\Gamma(W)|\propto\langle\Psi_1(W^*)|, \tag{52}$$

with normalization and phase chosen so that

$$\langle\bar\Gamma(W)|V|\Gamma(W)\rangle=1. \tag{53}$$

Here Ψ_1 is the eigenvector corresponding to the eigenvalue $\eta_1(W)$ of greatest modulus. With this choice the reduced interaction is

$$V_1(W)=V-\frac{V|\Psi_1(W)\rangle\langle\Psi_1(W^*)|V}{\langle\Psi_1(W^*)|V|\Psi_1(W)\rangle}, \tag{54}$$

and the coefficients $G_\nu(W)$ are, therefore, given by (47) as

$$G_\nu(W)=\begin{cases}1 & \nu=1\\ 0 & \nu>1.\end{cases} \tag{55}$$

The introduction of a quasiparticle with such vertices has the following advantages:

(a) The greatest eigenvalue $\eta_1(W)$ becomes zero for the reduced kernel $V_1(W)[W-H_0]^{-1}$, other eigenvalues remaining the same. [Use (33), (46), and (55).] Hence, if there originally is just one culpable eigenvalue outside the unit circle, the introduction of a quasiparticle according to (54) cures the divergence of the Born series. [If the original Born series diverges because several eigenvalues $\eta_\nu(W)$ are originally outside the unit circle, then the divergence can be cured by introducing an equal number of quasiparticles, with reduced interaction

$$V_1(W)=V-\sum_{|\eta_\nu|\geq1}\frac{V|\Psi_\nu(W^*)\rangle\langle\Psi_\nu(W)|V}{\langle\Psi_\nu(W^*)|V|\Psi_\nu(W)\rangle}. \tag{56}$$

This more general case can be easily handled, but we shall not discuss it further here.] If the Born series originally converges, then introducing the quasiparticle just improves the convergence.

(b) It could justly be remarked that the task of finding $|\Psi_1(W)\rangle$ is not much easier than the complete solution of the original Lippmann-Schwinger equation. But it must be realized that there is no necessity to construct $|\Gamma(W)\rangle$ and $\langle\bar\Gamma(W)|$ *exactly* according to (51)–(53). Assuming $\eta_1(W)$ to be the only original eigenvalue outside the unit circle, the divergence of the Born series will be cured as long as $G_1(W)$ is sufficiently close to unity, and the other $G_\nu(W)$ are sufficiently small. (Just what "sufficiently" means here depends partly on how far within the unit circle the nonculpable eigenvalues lie.) Furthermore, there is a direct and practical procedure for constructing a $|\Gamma\rangle$ and $\langle\bar\Gamma|$ for which the coefficients G_ν are arbitrarily close to the "ideal" values (55). Simply start with any initial trial $|\Gamma^{(0)}\rangle$ and $\langle\bar\Gamma^{(0)}|$, and calculate

$$|\Gamma(W)\rangle\propto[(W-H_0)^{-1}V]^M|\Gamma^{(0)}(W)\rangle, \tag{56}$$

$$\langle\bar\Gamma(W)|\propto\langle\bar\Gamma^{(0)}(W)|[V(W-H_0)^{-1}]^M. \tag{57}$$

For then the $G_\nu(W)$ will have the ν dependence

$$G_\nu(W)\propto\eta_\nu{}^{2M}(W)G_\nu{}^{(0)}(W), \tag{58}$$

and by choosing M large enough we can make $|G_1(W)|$ arbitrarily large compared with the other $|G_\nu(W)|$. Also, we should still normalize $|\Gamma\rangle$ and $\langle\bar\Gamma|$ according to

the prescription (53):

$$\langle\bar{\Gamma}(W)|V|\Gamma(W)\rangle=1. \tag{59}$$

For then according to (43)–(45) and (59):

$$\sum_\nu G_\nu(W)=1, \tag{60}$$

so that $G_1(W)$ will be close to one, and the other $G_\nu(W)$ much less.

(c) For W real and negative, the $V_1(W)$ given in Eq. (54) is Hermitian.[13] This is necessary if we want the theory modified by the introduction of the quasiparticle to *appear* acceptable by the usual standards of quantum mechanics. [But, of course, the modified theory is just a written version of the original one, and hence, automatically acceptable whether or not V_1 is Hermitian. In fact, Eq. (54) gives a non-Hermitian $V_1(W)$ for $W=E+i\epsilon$.]

(d) The propagator $\Delta(W)$ is given by (49) and (55) as

$$\Delta(W)=[1-\eta_1(W)]^{-1}. \tag{61}$$

But the $\eta_\nu(W)$ are real analytic functions in the cut W-plane (see Sec. VI) and, hence, so is $\Delta(W)$. Again, these analyticity and reality properties are necessary to preserve the *appearance* of quantum-mechanical consistency.

(e) If we choose the vertices exactly according to (51)–(53), then with the aid of (14) and (61) we can write Eq. (8) for $T(W)$ as

$$T(W)=T_1(W)+\frac{V|\Psi_1(W)\rangle\langle\Psi_1(W^*)|V}{\langle\Psi_1(W^*)|V|\Psi_1(W)\rangle}\frac{1}{1-\eta_1(W)}. \tag{62}$$

Continuing in this way we could, if we wished, develop $T(W)$ as a series of separable terms like that in (62).

The rest of this section is devoted to a proof that (55) represents the best possible set of $G_\nu(W)$ that we can have for $W<0$ if we want to retain the Hermiticity of $V_1(W)$. Requiring $\Delta(W)$ to be analytic in the cut W plane then forces us to (55) for all W.

We begin by showing that if $V_1(W)$ is to be Hermitian for $W<0$, then all of the $G_\nu(W)$ must be real, and all of the $G_\nu(W)\eta_\nu(W)$ have the same sign. [The reality of $\eta_\nu(W)$ for $W<0$ is shown in Sec. VI.] If $V_1(W)$ is Hermitian we must have

$$\langle\bar{\Gamma}(W)|=\xi(W)\langle\Gamma(W)|, \tag{63}$$

where $\xi(W)$ is real. It follows then from (47) and (14) that

$$G_\nu(W)\eta_\nu(W)=\xi(W)\frac{|\langle\Psi_\nu(W)|V|\Gamma(W)\rangle|^2}{\langle\Psi_\nu(W)|\{W-H_0\}|\Psi_\nu(W)\rangle}. \tag{64}$$

So since $W-H_0$ is a negative-definite operator for $W<0$, the $G_\nu\eta_\nu$ must be real, and all must be of opposite sign to $\xi(W)$.

Now, the new eigenvalues $\chi(W)$ of the reduced kernel are given by (33) as the zeroes of $\Delta(W,\chi^{-1})$, where the propagator $\Delta(W,\lambda)$ is given by (46). With all $G_\nu\eta_\nu$ real and of the same sign, this function has the properties:

(A) When χ is complex or real so is Δ, since

$$\mathrm{Im}\Delta(W,\chi^{-1})=-(\mathrm{Im}\chi)\sum_\nu\frac{G_\nu(W)\eta_\nu(W)}{|\chi-\eta_\nu(W)|^2}. \tag{65}$$

All terms in the sum have the same sign, and, hence, can not cancel.

(B) For real χ, $\Delta(W,\chi^{-1})$ either increases (for $\xi>0$) or decreases (for $\xi<0$) monotonically with χ, since

$$\frac{d}{d\chi}\Delta(W,\chi^{-1})=-\sum_\nu\frac{G_\nu(W)\eta_\nu(W)}{[\chi-\eta_\nu(W)]^2}. \tag{66}$$

All terms in the sum have the same sign. Between adjacent η_ν with $G_\nu\neq0$, Δ either rises from $-\infty$ to $+\infty$ or drops from $+\infty$ to $-\infty$.

From (A) we see that for $W<0$, all zeroes $\chi(W)$ of $\Delta(W,\chi^{-1})$ must be real. A typical function $\Delta(\chi^{-1})$ with property (B) is shown in Fig. 1 for real χ, in the case where $\xi<0$, and where there is just one $\eta_1>1$ and no $\eta_\nu<-1$. We see that the subtraction in Eq. (54) pulls all eigenvalues to the left, lowering the culpable eigenvalue η_1. (Had we chosen $\xi>0$, all eigenvalues would be pulled to the right, increasing the culpable eigenvalue η_1, and worsening the divergence.)

The new eigenvalues χ all lie below the corresponding η's, but above the next lower η. Clearly, the best we can do is to lower all of the positive eigenvalues down all the way to the next lower η, and to lower the negative eigenvalues not at all. (The solid lines in Fig. 1 show a choice which almost accomplishes this, and which does succeed in making the Born series converge.) In order to accomplish this aim exactly, we must succeed in making each η_ν except η_1 an eigenvalue $\chi_{\nu-1}$ of the reduced kernel. This requires that all G_ν except G_1 are zero. In this case the single remaining eigenvalue χ_∞ of the reduced kernel is given by (33) as

$$\chi_\infty(W)=\eta_1(W)[1-G_1(W)]. \tag{67}$$

So we see that (51)–(53) does represent the ideal choice of bare vertices.

Actually the speed of convergence of the Born series (12) for $T_1(W)$ depends only on the magnitude of the greatest χ. (This is just a guess, and certainly not always true.) Since the choice $G_2=0$ ensures that the next-to-greatest original eigenvalue η_2 is an eigenvalue χ_1 of the

[13] R. Blankenbecler has pointed out in a private communication that if V is approximated by $V_S=V|\Gamma\rangle\langle\Gamma|V$, and if V_S is Hermitian and normalized according to (59), then an error of *known* sign is made in phase shifts and binding energies. This point is now being explored by R. Blankenbecler and M. Sugar. It would be interesting to see whether a similar result holds for our "ideal" choice of $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$, which does not give an Hermitian V_S in scattering problems. However, whether or not the error is of known sign, it may always be reduced as much as we like by using (8) to (11) with the series (12).

reduced kernel, there is nothing particular gained by arranging that χ_∞ be zero. The reduced Born series will converge equally well for any G_1 such that $|\chi_\infty|<|\eta_2|$, or, in other words, for any G_1 such that

$$|1-G_1(W)|\leqq|\eta_2(W)/\eta_1(W)|. \tag{68}$$

But the choice $G_1=1$ is particularly convenient, since we can always use the procedure outlined above under (b) to get sufficiently close to (55) for the reduced Born series to converge.

VI. EIGENVALUES OF THE KERNEL: BEHAVIOR AND INTERPRETATION

We have seen that any discussion of the convergence of the Born series must center upon the eigenvalues $\eta_\nu(W)$ and the eigenvectors $|\Psi_\nu(W)\rangle$ of the kernel $[W-H_0]^{-1}V$. For the Born series diverges whenever some eigenvalue $\eta_\nu(W)$ lies outside the unit circle, and the divergence can be cured by introducing a quasi-particle with bare vertices not too different from the "ideal" choice (51)–(53), and hence, approximating the eigenvector $\Psi_\nu(W)$.

In order to facilitate the diagnosis and cure of the divergence of the Born series, we shall, therefore, first discuss the physical interpretation of the $\eta_\nu(W)$ and $\Psi_\nu(W)$, and then treat some of their general properties. A few particular examples are given in the next section.

The eigenvalue problem is

$$[W-H_0]^{-1}V|\Psi_\nu(W)\rangle=\eta_\nu(W)|\Psi_\nu(W)\rangle. \tag{14}$$

Here W can be anywhere in the complex plane, except on the positive real axis. [In scattering problems we deal with $W=E+i\epsilon$, with $0<\epsilon\ll E$. See, e.g., Eq. (4).] This can be rewritten as a modified Schroedinger equation,

$$[H_0+\eta_\nu^{-1}(W)V]|\Psi_\nu(W)\rangle=W|\Psi_\nu(W)\rangle. \tag{69}$$

It is understood that $|\Psi_\nu(W)\rangle$ must be normalizable; $\eta_\nu(W)$ is determined by the condition that such a solution exist. Of course, if the "interaction" $\eta_\nu^{-1}V$ is Hermitian then such "bound states" can only exist for real $W<0$; hence, all $\eta_\nu(W)$ must be complex for W complex. [In particular, $\eta_\nu(W)$ is always complex in scattering problems; $|\Psi_\nu(E+i\epsilon)\rangle$ is normalizable for finite ϵ, but not for $\epsilon=0$.]

Equation (69) allows us to describe $\eta_\nu(W)$ as a number by which the interaction V may be divided in order to have a bound state at energy W. We, therefore, conclude that the Born series (5) at energy W will converge if and only if there exists no bound state at W, not only for the actual interaction V but also for *any weaker* interaction of form V/η, with $|\eta|\geqq1$. If V is a repulsive interaction, or if W is complex or positive, then there can be no actual bound state at energy W, but the Born series may still diverge because of some bound state that *would* be at W if the interaction were V/η, with $|\eta|\geqq1$.

When $\eta_\nu(W)=1$ for some $W<0$, then there is a bound state with binding energy $-W$ for the actual interaction V. When

$$\mathrm{Re}\eta_\nu(E_0+i\epsilon)\cong1,\quad \mathrm{Im}\eta_\nu(E_0+i\epsilon)\ll1, \tag{70}$$

for some $E_0>0$, then there is an energy eigenstate with energy $+E_0$ for a non-Hermitian interaction $V/\eta(E_0+i\epsilon)$ which is *almost* the actual interaction. In this case the propagator $\Delta(E_0+i\epsilon)$ given by Eq. (42) or (49) (and, hence, also the S matrix), becomes very large at energy E_0. We call this a resonance or a virtual state, depending on whether $\mathrm{Re}\eta(E+i\epsilon)$ passes one as E passes E_0, or just approaches it and then recedes. We shortly see that in potential problems, the wave functions ψ_ν contain only outgoing waves, agreeing with the intuitive idea of a resonance as a decaying particle. The behavior of the phase shift is discussed later.

A special case of particular interest is that of a spinless nonrelativistic particle of mass m moving with orbital angular momentum l in a local central short-range potential $V(r)$. In this case (69) becomes

$$\left[-\frac{1}{2m}\frac{d^2}{dr^2}+\frac{l(l+1)}{2mr^2}+\frac{V(r)}{\eta_\nu(W)}\right]\psi_\nu(r;W)=W\psi_\nu(r;W), \tag{71}$$

and the condition that $|\Psi_\nu\rangle$ be normalizable yields

$$\psi_\nu(r;W)\sim r^{l+1},\quad r\to0, \tag{72}$$

$$\sim e^{ikr},\quad r\to\infty, \tag{73}$$

where k is determined by

$$W=k^2/2m;\quad \mathrm{Im}k>0. \tag{74}$$

(We omit the index l everywhere.) "Short range" is taken to mean that for some $a>0$

$$\int_0^\infty r|V(r)|e^{r/a}dr<\infty. \tag{75}$$

The "range" is the greatest lower bound of a satisfying (75).

For $W<0$ we get $k=i\kappa$ with $\kappa>0$, and so $\psi_\nu(r;W)$ has the correct behavior as $r\to\infty$ for a bound state wave function. For $W=E+i\epsilon$ with $E>0$, k becomes real and *positive*, and so $\psi_\nu(r,k)$ is a wave function with outgoing waves only. Of course such solutions never exist for Hermitian potentials, but they can and do exist for the complex potentials $V(r)/\eta_\nu(W)$.

The wave function $\psi(r,k)$ for a potential $\lambda V(r)$ is given by

$$\psi(r,k,\lambda)\propto f(-k,\lambda)f(r,k,\lambda)-f(k,\lambda)f(r,-k,\lambda), \tag{76}$$

where $f(k,\lambda)$ is the usual Jost function,[14] and $f(r,k,\lambda)$ is that solution of Schrödinger's equation which approaches $\exp(-ikr)$ as $r\to\infty$. Hence, we see that the eigenvalues $\eta_\nu(W)$ for potential λV may be defined as

[14] For general l, see R. Newton (reference 10).

the roots of the equation

$$f(-k, \lambda/\eta)=0. \tag{77}$$

This will be useful when we come to discuss the behavior of the $\eta_\nu(W)$ in potential theories.

We shall now describe the properties of $\eta_\nu(W)$ for a general interaction V. No attempt will be made at extreme rigor, but our conclusions are plausible, and are borne out by the examples presented in the next section.

We begin by recalling from Sec. III that for W not real and positive, the $\eta_\nu^{-1}(W)$ are the poles of a meromorphic function, the resolvent $T(W,\lambda)$. It follows immediately that for such W:

(A) The $\eta_\nu(W)$ form a discrete set.

(B) At most a finite number $N(W)$ of eigenvalues $\eta_\nu(W)$ lie outside the unit circle. For if an infinite number of the $\eta_\nu(W)$ had $|\eta_\nu(W)|>1$, then the $\eta_\nu^{-1}(W)$ would have to have a limit point at some finite λ within the unit circle; this is impossible for the poles of a meromorphic function. [Both (A) and (B) follow also from the fact that the $\eta_\nu^{-1}(W)$ are zeroes of an integral function, the modified Fredholm determinant.]

Result (B) is vital to our program, since it shows that the introduction of a *finite* number $\mathfrak{N}(W)$ of fictitious elementary particles can always cure the divergence of the Born series at a fixed W. [However, we see in Sec. VII that $\mathfrak{N}(W)$ can only be expected to be a bounded function of W for short-range forces.] Alternate proofs of (B) will be given as we go along.

We shall next study the $\eta_\nu(W)$ for negative real W; the results obtained will then be extended to all W by analytic continuation. The $\eta_\nu(W)$ are the eigenvalues of an operator $K(W)$:

$$K(W)=-(H_0-W)^{-1/2}V(H_0-W)^{-1/2}, \tag{78}$$

which for $W<0$ is Hermitian. It follows that

(C) The $\eta_\nu(W)$ for $W<0$ are all real. Also, since the trace of $K(W)K^\dagger(W)$ is finite (see Appendix B), we have

(D) Then $\eta_\nu(W)$ for $W<0$ obey the sum rules

$$\sum_\nu \eta_\nu^N(W)=\tau_N(W), \quad (N=2, 3, \cdots), \tag{79}$$

where $\tau_N(W)$ is finite, and given by

$$\tau_N(W)=\mathrm{Tr}\{K^N(W)\},$$

or using (78)

$$\tau_N(W)=\mathrm{Tr}\{[(W-H_0)^{-1}V]^N\}. \tag{80}$$

This shows that as $\nu\to\infty$

$$\nu^{1/2}\eta_\nu(W)\to 0, \tag{81}$$

which provides an independent proof of (B). We shall now prove the more surprising property,

(E) Each $\eta_\nu(W)$ for $W<0$ is either positive and increasing or negative and decreasing over the whole range from $W=-\infty$ to $W=0$. These two cases will be called "attractive" and "repulsive" eigenvalues, respectively, for reasons which will be made clear under (F). To see that these two possibilities are, in fact, exhaustive, we note that for $W<0$, (69) gives

$$[(H_0-W)\eta_\nu(W)+V]|\Psi_\nu(W)\rangle=0,$$

and

$$\langle\Psi_\nu(W)|[(H_0-W)\eta_\nu(W)+V]=0,$$

so that

$$\begin{aligned}0&=\frac{d}{dW}\langle\Psi_\nu(W)|[(H_0-W)\eta_\nu(W)+V]|\Psi_\nu(W)\rangle\\&=\langle\Psi_\nu(W)|\left[\frac{d}{dW}(H_0-W)\eta_\nu(W)\right]|\Psi_\nu(W)\rangle.\end{aligned}$$

Hence,

$$\frac{1}{\eta_\nu(W)}\frac{d\eta_\nu(W)}{dW}=\frac{\langle\Psi_\nu(W)|\Psi_\nu(W)\rangle}{\langle\Psi_\nu(W)|(H_0-W)|\Psi_\nu(W)\rangle}. \tag{82}$$

But H_0-W is a positive-definite operator for $W<0$, and so

$$\frac{1}{|W|}>\frac{1}{\eta_\nu(W)}\frac{d\eta_\nu(W)}{dW}>0. \tag{83}$$

If $\pm\eta_\nu(W)$ is positive at any point $W_1<0$, then (83) tells us that it will increase for all $W>W_1$, and of course, therefore, stay positive. Letting $W_1\to-\infty$, we get (E).

We have already noted that bound states occur when one of the η_ν becomes unity. [See Eq. (50).] So (E) tells us that only the attractive eigenvalues yield bound states, and then only when

$$\eta_\nu(0)\geqq 1. \tag{84}$$

There is just one bound state for each eigenvalue $\eta_\nu(W)$ satisfying (84). The fact that such eigenvalues increase with W just corresponds to the fact that any bound state is always deepened by strengthening the interaction.

A general interaction V will have both repulsive and attractive eigenvalues. But things are simpler in special cases.

(F) A purely attractive or repulsive interaction V has only attractive or repulsive eigenvalues. For (14) gives

$$\eta_\nu(W)=\frac{\langle\Psi_\nu(W)|V(W-H_0)^{-1}V|\Psi_\nu(W)\rangle}{\langle\Psi_\nu(W)|V|\Psi_\nu(W)\rangle}, \tag{85}$$

and so for $W<0$ the sign of $\eta_\nu(W)$ is always the same as that of

$$-\langle\Psi_\nu(W)|V|\Psi_\nu(W)\rangle. \tag{86}$$

The fact that all eigenvalues are repulsive for a repulsive interaction tells us the obvious fact that no repulsive interaction can ever yield a bound state.

We shall now leave the real axis and treat the $\eta_\nu(W)$ for complex W.

(G) Each $\eta_\nu(W)$ is analytic in the complex W plane, cut along the real axis from $W=0$ to $W=\infty$. This is apparent upon inspection of Eq. (14). The only exception expected would arise if several $\eta_\nu(W)$ became equal at some W; we assume this doesn't happen.

By analytic continuation of (C) and (D), we see that

(H) The $\eta_\nu(W)$ satisfy

$$\eta_\nu^*(W)=\eta_\nu(W^*). \tag{87}$$

Equation (88) follows from (C), (G), and the Schwarz reflection principle. It can also be derived using time reversal rather than analyticity arguments. If we apply the anti-unitary time reversal operator θ to (14), we get

$$[W^*-H_0]^{-1}V\theta|\Psi_\nu(W)\rangle=\eta_\nu^*(W)\theta|\Psi_\nu(W)\rangle. \tag{88}$$

This yields (87), and also

$$\theta|\Psi_\nu(W)\rangle=|\Psi_\nu(W^*)\rangle. \tag{89}$$

(I) The $\eta_\nu(W)$ obey the sum rule (79) for complex W as well as for $W<0$.

(J) An attractive (repulsive) eigenvalue $\eta_\nu(W)$ has positive-definite (negative-definite) imaginary part in the upper half of the complex W plane. For we have already remarked that $\mathrm{Im}\eta_\nu(W)$ never vanishes for complex W, since when $\eta_\nu(W)$ is real the operator $H_0+\eta_\nu^{-1}(W)V$ is Hermitian, and hence, can only have real eigenvalues W. So the imaginary part of $\eta_\nu(W)$ stays positive-definite or negative-definite throughout the region $\mathrm{Im}W>0$. But at a point $W=W_0+i\delta$ (where $W_0<0$ and $0<\delta\ll|W_0|$) we have

$$\eta_\nu(W_0+i\delta)=\eta_\nu(W_0)+i\delta\eta_\nu'(W_0).$$

The derivative $\eta_\nu'(W_0)$ is positive definite (or negative definite) for an attractive (or repulsive) eigenvalue, so $\eta_\nu(W)$ has positive-definite (or negative-definite) imaginary part at $W_0+i\delta$, and, hence, everywhere in the upper half W plane.

(K) The $\eta_\nu(W)$ vanish in the limit $|W|\to\infty$, and have the spectral representation

$$\eta_\nu(W)=\int_0^\infty \frac{\rho_\nu(E)}{E-W}dE. \tag{90}$$

Here $\rho_\nu(E)$ is real, and positive (negative) when η_ν is an attractive (repulsive) eigenvalue. The vanishing of $\eta_\nu(W)$ as $W\to-\infty$ follows from (I); if $\tau_2(W)$ exists at all it clearly must vanish as $W\to-\infty$. The spectral representation follows then rigorously by use of (G), (J), and the Herglotz theorem.[15] One consequence of the vanishing of $\eta_\nu(+\infty+i\epsilon)$ is that the Born series always must converge at sufficiently high energy.[16]

Nothing else can be said about the individual $\eta_\nu(W)$ without putting further restrictions on the interaction V. This is demonstrated clearly by the first example presented in the next section, which shows that for *any* given $\eta_1(W)$ having the above properties, an interaction V may be constructed which yields $\eta_1(W)$ as one of the eigenvalues.

But in the problem of scattering by a short-range potential [as specified by (71)–(75)] it is possible to say considerably more about the $\eta_\nu(W)$.

(L) The function $\eta_\nu(W)$ has only one singularity in the complex W plane, a branch point of the $\sqrt{W}$ variety at $W=0$. For it is obvious from (77) that the singularities of the η_ν when written as a function of $k=(2mW)^{1/2}$ must be the same as those of the Jost function $f(-k,\lambda)$. Hence, $\eta_\nu(k)$ may be extended from $\mathrm{Im}k>0$ (the physical W sheet) to the larger region[14]

$$\mathrm{Im}k>-1/(2a), \tag{91}$$

and so $\eta_\nu(W)$ is analytic in $\sqrt{W}$ on the physical sheet. (We will see in the next section that the same result holds for the Coulomb potential, except that a pole W^{-1} is superimposed on the branch singularity $\sqrt{W}$.)

(M) For $E\to0$ the spectral function of $\eta_\nu(W)$ has the behavior

$$\rho_\nu(E)\to C_\nu E^{l+\frac{1}{2}}. \tag{92}$$

This can be seen by noting from (90) and (85) that

$$\pi\rho_\nu(E)=\mathrm{Im}\eta_\nu(E+i\epsilon)=-\frac{|\langle E|V|\Psi_\nu(E+i\epsilon)\rangle|^2}{\langle\Psi_\nu(E+i\epsilon)|V|\Psi_\nu(E+i\epsilon)\rangle}. \tag{93}$$

In potential theory this becomes

$$\rho_\nu(E)=\frac{-2mk}{\pi^2}\left[\left|\int_0^\infty j_l(kr)V(r)\psi_\nu(r;k)rdr\right|^2\Big/\int_0^\infty V(r)|\psi_\nu(r;k)|^2dr\right], \tag{94}$$

and the result (92) follows upon use of (72) and (75).

It follows from (90) and (92) that for small E

$$\eta_\nu(E+i\epsilon)-\eta_\nu(0)=B_\nu E+i\pi C_\nu E^{l+\frac{1}{2}}, \tag{95}$$

where B_ν and C_ν are real. So (95) tells us that this difference is essentially pure imaginary and grows as $\sqrt{E}$ for S-wave scattering, while for higher partial waves it grows as E and is essentially pure real. (See Figs. 4 and 5.)

(N) The sum rule (79) holds for all W even with $N=1$; in this case it says that

$$\sum_\nu \eta_\nu(W)=-2imk\int_0^\infty r^2V(r)j_l(kr)h_l^{(1)}(kr)dr. \tag{96}$$

[15] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, 1943), p. 23. Without using (I), the Herglotz theorem alone would show that $\eta_\nu(W)=O(W)$ for $W\to\infty$.

[16] C. Zemach and A. Klein, Nuovo Cimento **10**, 1078 (1958); A. Klein and C. Zemach, Ann. Phys. (N. Y.) **7**, 440 (1959); W. Kohn, reference 5.

The convergence of the integral is guaranteed by (75), so we can get (96) for $W<0$ by the same arguments that gave (79), and for all W by analytic continuation.

(O) The phase shift for the l-th partial wave at energy E is

$$\delta(E)=\sum_\nu \delta_{(\nu)}(E), \tag{97}$$

where the "elemental phase shift" $\delta_{(\nu)}(E)$ is defined by

$$\delta_{(\nu)}(E)=-\arg(1-\eta_\nu(E+i\epsilon)). \tag{98}$$

This follows immediately from Eq. (172) for the Fredholm determinant:

$$D(W)=\prod_\nu [1-\eta_\nu(W)],$$

if we recall that $\delta(E)$ is just the phase of $D^*(E+i\epsilon)$. The convergence of (172) and (97) is ensured by (96). [Actually, the convergence seems to be quite rapid. The first few $\delta_{(\nu)}(E)$ for S-wave scattering by the Hulthén potential are shown in Fig. 4, for a particular energy E and coupling parameter λ. We see in this case that $\delta_{(1)}=127°$, $\delta_{(2)}=13°$, $\delta_{(3)}=3°$, and $\delta_{(4)}=1°$, so $\delta=144°$.]

(P) For an attractive eigenvalue $\eta_\nu(W)$, the corresponding elemental phase shift stays in the range

$$0\leqq\delta_{(\nu)}(E)\leqq\pi, \tag{99}$$

while for a repulsive eigenvalue we have instead

$$-\pi\leqq\delta_{(\nu)}(E)\leqq 0. \tag{100}$$

This follows trivially from (J) and the definition (98). For a purely attractive (repulsive) potential $V(r)$ this shows that the full phase shift $\delta(E)$ is positive (negative) for all E.

Other results dealing with the high- and low-energy behavior of the elemental and full-phase shift are presented in Sec. VIII, and are used there to give a new proof of Levinson's theorem.

(Q) The $\eta_\nu(W)$ can be determined from the elemental phase shifts $\delta_{(\nu)}(E)$ by the formula

$$1-\eta_\nu(W)=\exp\left\{-\frac{1}{\pi}\int_0^\infty \frac{\delta_{(\nu)}(E)}{E-W}dE\right\}. \tag{101}$$

If there is a bound state at energy $-B$ this must be modified to read

$$1-\eta_\nu(W)=\left(1+\frac{B}{W}\right)\exp\left\{-\frac{1}{\pi}\int_0^\infty \frac{\delta_{(\nu)}(E)}{E-W}dE\right\}. \tag{102}$$

These formulas can be derived in the same manner as the more familiar formulas relating the full Fredholm determinant to the full phase shift. [The simplest procedure is to apply the Herglotz theorem[15] to the functions

$$\ln[1-\eta_\nu(W)] \quad \text{or} \quad \ln\left\{\frac{1-\eta_\nu(W)}{1+B/W}\right\},$$

and use (98) and (90).]

(R) At most, a finite number of eigenvalues $\eta_\nu(W)$ ever leave the unit circle for *any* W. For we have already seen in (B) that the number of $|\eta_\nu|>1$ is a finite number $\mathfrak{N}(W)$, except perhaps for real positive W. But then (L) and (M) shows that $\mathfrak{N}(W)$ is finite everywhere, and bounded uniformly in W.

This result is important to all our future work, since it shows that perturbation theory can be made to work at all energies in strong coupling problems by the introduction of a *fixed* number of quasiparticles.

We close this section by summarizing the behavior of the $\eta_\nu(W)$ as W rises along the real axis from $-\infty$ to 0, and then continues just above the real axis, from $0+i\epsilon$ to $\infty+i\epsilon$.

Any attractive eigenvalue $\eta_\nu(W)$ rises monotonically from zero at $W=-\infty$ to some finite positive value $\eta_\nu(0)$ at $W=0$. Then as W increases further (with $W=E+i\epsilon$, $E>0$) $\eta_\nu(W)$ becomes complex, with positive imaginary part. (This rise into the complex plane is initially vertical for S-wave scattering by a short-range potential, and horizontal for higher partial waves.) Eventually for $E\to\infty$, $\eta_\nu(E+i\epsilon)$ returns through the upper half of the complex plane to the origin. (This behavior is shown graphically for special cases in Figs. 4 and 5.) When some η trajectory leaves the unit circle, we have a bound state (for $W<0$) or a resonance (for $W=E+i\epsilon$, $E>0$). In the latter case the elemental phase shift is increasing through an angle less than $\pi/2$; if (70) holds the resonance will be narrow and the angle almost $\pi/2$. The Born series begins to diverge at the lowest such energy, and stays divergent until the last η trajectory re-enters the unit circle. The elemental phase shift is then less than $\pi/2$.

We have noted in this case that if $\eta_\nu(0)>1$, then there is one bound state for this trajectory. If $\eta_\nu(0)<1$ the bound state is absent, but if $\eta_\nu(0)$ is only slightly less than one then the η trajectory will still leave the unit circle, giving a low-energy resonance. This conclusion does not hold for S-wave scattering, since there $\eta_\nu(E+i\epsilon)$ rises vertically for small E, and need not intersect the unit circle. In fact, it is well known that a purely attractive potential never does yield S-wave resonances.

The same behavior holds for $-\eta_\nu(W)$ in the case of a repulsive eigenvalue. However, the intersection of an η trajectory with the unit circle is not associated here with an actual bound state or resonance, but instead with one that would exist if the interaction were $-V$.

VII. EIGENVALUES OF THE KERNEL: EXAMPLES

The simplest possible example is that of a separable (and hence nonlocal) interaction,

$$V=-\lambda|f\rangle\langle f|,$$
$$|f\rangle\equiv\sum_n\int_0^\infty dE f_n(E)|En\rangle. \tag{103}$$

In this case there is an eigenvector

$$|\Psi_1(W)\rangle=\int_0^\infty \frac{dE}{W-E}\sum_n f_n(E)|En\rangle \qquad (104)$$

with eigenvalue

$$\eta_1(W)=\int_0^\infty \frac{\rho_1(E)}{E-W}dE, \qquad (105)$$

$$\rho_1(E)=\lambda\sum_n |f_n(E)|^2, \qquad (106)$$

and elemental phase shift

$$\delta_{(1)}(E)=\arctan\left\{\pi\rho_1(E)\Big/1-\mathrm{P}\int_0^\infty \frac{\rho_1(E')}{E-E'}dE'\right\}. \qquad (107)$$

All other eigenvalues are zero, so this is the physical phase shift.

It is easy to check that (A)–(K) and (N)–(R) are all correct in this case. However, (L) and (M) need not hold. By fiddling with the functions $f_n(E)$, we can make $\rho_1(E)$ virtually anything we like. In particular, the cut from $W=0$ to $W=\infty$ can be made a natural boundary rather than a branch line.

All other examples that we treat will deal with local potentials. The simplest such is the Coulomb potential

$$V(r)=-Ze^2/r. \qquad (108)$$

Here $\eta_\nu(W)$ is determined by (71)–(73), except that we allow the exponential in (73) to be multiplied by a polynomial in kr. The exact solution for $\psi_\nu(r)$ is familiar:

$$\psi_\nu(r;W)=r^{l+1}L_{\nu+l}^{2l+1}(-ikr)e^{ikr}, \qquad (109)$$

where L is the usual Laguerre polynomial, and ν is an integer greater than l. For this to actually satisfy (71), we must have

$$\eta_\nu(W)=iZe^2m/\nu k=Ze^2/2\nu(-2W/m)^{1/2}. \qquad (110)$$

It is easy to check that (A)–(K) hold here, but not (L)–(R); the spectral function $\rho_\nu(E)$ in Eq. (90) is

$$\rho_\nu(E)=Ze^2m/\nu k. \qquad (111)$$

As $\nu\to\infty$, $\eta_\nu\to 0$, but too slowly for (96) and (97) to converge. Instead, we have

$$\delta(E)=\frac{\gamma mZe^2}{k}-\sum_{\nu=l+1}^{\infty}\arg\{[1-\eta_\nu(W)]e^{\eta_\nu(W)}\},$$

where $\gamma=0.577\cdots$. Also $\eta_\nu(0)$ is infinite, so that (M) and (R) cannot hold here. Clearly, the failure of (M)–(S) for the Coulomb potential is wholely due to its long range.

The $|\eta_\nu(W)|$ for the Coulomb potential are plotted against W in Fig. 2, for the case $Z=+1$. The $\eta_\nu(W)$ are positive for $W<0$, and so the intersections of these curves with the vertical line $|\eta|=1$ determine the bound

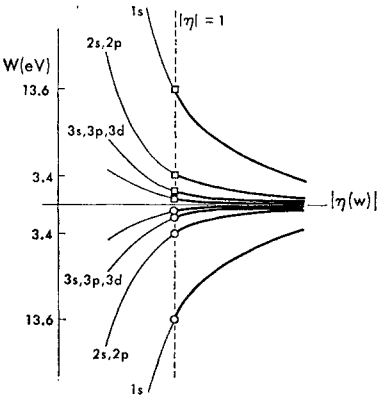

FIG. 2. Absolute values of the eigenvalues $\eta_\nu(W)$ vs W, for the Coulomb potential $-e^2/r$. The Born series begins to diverge at the lowest bound state (small circles) and continues to diverge (darkened curves) until all $\eta_\nu(W)$ have re-entered the unit circle (small squares). For $W<0$ the $\eta_\nu(W)$ are positive, while for $W>0$ the phase is $+i$.

state energies,

$$k_\nu=iZe^2m/\nu, \qquad (112)$$

which yield the not unfamiliar formula,

$$W_\nu=-mZ^2e^4/2\nu^2. \qquad (113)$$

The Born series for $T(W)$ diverges for a given partial wave l as long as one of the $|\eta_\nu(W)|$ with $\nu\geqq l+1$ lies to the right of the line $\eta=1$. Hence, the S-wave series diverges from $W=-13.6$ eV to $W=13.6$ eV$+i\epsilon$; the P-wave series diverges from $W=-3.4$ eV to $W=3.4$ eV$+i\epsilon$, and so on. (When we refer to the Born series here, it should be understood that we use it to calculate $\langle E|T(E+i\epsilon)|E'\rangle$ in each partial wave, then sum over l, and only then set $E=E'$.)

But the important point is that these divergences can be cured. For example, if we introduce a fictitious spinless elementary particle corresponding to the $1s$ bound state, then the $l=0$ Born series will be rendered convergent, except in the energy range -3.4 eV to $+3.4$ eV$+i\epsilon$. The Columb potential has the unfortunate property that every η trajectory gets outside the unit circle for some small W, and hence, if we wanted to make the Born series convergent for *all* W, we would have to introduce an infinite number of fictitious elementary particles.

However, this difficulty is happily absent for short-range potentials. To illustrate this, we consider a final example, the S-wave scattering by a Hulthén potential[17]:

$$V(r)=-\frac{\lambda}{2ma^2}[e^{r/a}-1]^{-1}. \qquad (114)$$

This looks like the Coulomb potential (108) for $r\ll a$, if we identify

$$Ze^2=\lambda/2ma \qquad (115)$$

[17] L. Hulthén, Ark. Mat. Astron. Fysik **28A**, No. 5 (1942); **29B**, No. 1 (1942).

but for $r \gg a$ it is cut off, approaching the exponential potential

$$V(r) \cong -\frac{\lambda}{2ma^2} \exp(-r/a). \tag{116}$$

It is well known[18] that the exact Jost function here is

$$f(k,\lambda) = \prod_{\nu=1}^{\infty} \left[1 - \frac{\lambda}{\nu(\nu+2ika)} \right], \tag{117}$$

and so the roots of Eq. (77) are

$$\eta_\nu(W) = \frac{\lambda}{\nu(\nu-2ika)} \quad (\nu \geqq 1)$$
$$= \frac{\lambda}{\nu(\nu+[-8mWa^2]^{1/2})}. \tag{118}$$

For example, the *exact* solution of (71) for $\nu = 1$ is

$$\psi_1(r,k) = e^{ikr}(1-e^{-r/a}). \tag{119}$$

The spectral function of the $\eta_\nu(W)$ is

$$\rho_\nu(E) = \frac{\lambda(8mEa^2)^{1/2}}{\pi\nu(\nu^2+8mEa^2)}, \tag{120}$$

and the elemental phase shifts are

$$\tan\delta_{(\nu)}(E) = \frac{2\lambda ka}{\nu(\nu^2-\lambda+4k^2a^2)}. \tag{121}$$

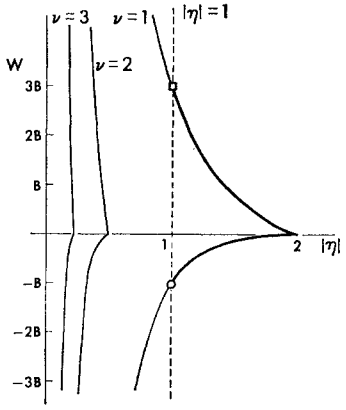

FIG. 3. Absolute values of the first three S-wave eigenvalues $\eta_\nu(W)$ vs W, for the Hulthén potential

$$V(r) = -\frac{\lambda}{2ma^2}[e^{r/a}-1]^{-1}.$$

[Here B is the energy $(8ma^2)^{-1}$.] We have drawn these curves for $\lambda = 2$, but the η_ν just scale with λ. The Born series begins to diverge at the bound state (small circle) and continues to diverge (darkened line) until $\eta_1(W)$ has re-entered the unit circle (small square). For $W<0$ the $\eta_\nu(W)$ are positive, while for $W>0$ they are complex.

[18] See reference 4.

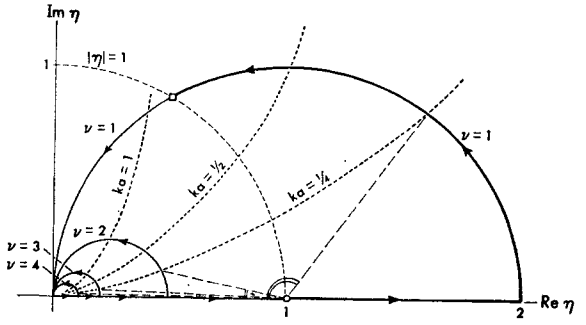

FIG. 4. Trajectories of the first four S-wave eigenvalues $\eta_\nu(W)$ in the complex plane, for the Hulthén potential

$$V(r) = -\frac{\lambda}{2ma^2}[e^{r/a}-1]^{-1}.$$

[Here k as usual is defined as $(2mW)^{1/2}$.] We have drawn these curves for $\lambda = 2$, but the $\eta_\nu(W)$ just scale with λ. The arrows indicate direction of the trajectories as W increases from $-\infty$ to $\infty + i\epsilon$. The Born series begins to diverge at the bound state (small circle) and continues to diverge (darkened trajectory) until $\eta_1(W)$ re-enters the unit circle (small square). The angles shown by small double arcs near $\eta = 1$ are the first few "elemental" phase shifts $\delta_{(\nu)}(W)$ at $ka = \frac{1}{4}$; we see that $\delta_{(1)}$ is somewhat less than π (actually 127°) while the other $\delta_{(\nu)}$ are close to zero. {This geometrical construction was suggested by some work of C. Lovelace [Nuovo Cimento **25**, 730 (1962)] on Regge poles.}

For $ka \gg \nu$ these formulas revert to the Coulomb ones. For finite a, we may easily check that *all* of the properties (A)–(R) hold here.

We have plotted $|\eta_\nu(W)|$ versus in Fig. 3. The number of η-trajectories that leave the unit circle at any energy is $\{|\lambda|^{1/2}\}$, and hence, this is the precise number of fictitious elementary particles that must be introduced if we wish to render the S-wave Born series convergent at all energies.

We have also plotted in Fig. 4 the trajectory of the complex numbers $\eta_\nu(W)$ for W rising from $-\infty$ to 0, and then becoming $W = E + i\epsilon$, with E rising from 0 to $+\infty$, in the special case $\lambda = 2$. The dark lines indicate the part of the η trajectory on which the Born series diverges; in this case the series diverges from $W = -B = -(8ma^2)^{-1}$ (the bound-state energy) to $W = 3B + i\epsilon$. No resonances can occur for the S-wave, so we have plotted in Fig. 5 a guess at what the η-trajectories look like for $l = 1$. Again, the Born series diverges along the dark lines.

VIII. LEVINSON'S THEOREM

We have studied the eigenvalues $\eta_\nu(W)$ in order to gain insight into the failure and repair of perturbation theory. Unexpectedly, the information assembled in Sec. VI turns out to be sufficient to provide a new proof of Levinson's theorem. We present it here not because it is a better proof, but because it makes clearer the relation between the behavior of phase shifts and the convergence of the Born series.

It has been shown in Sec. VI that the phase shift $\delta(E)$ for scattering by a short-range potential in the l-th partial wave is the sum of "elemental" phase-shifts

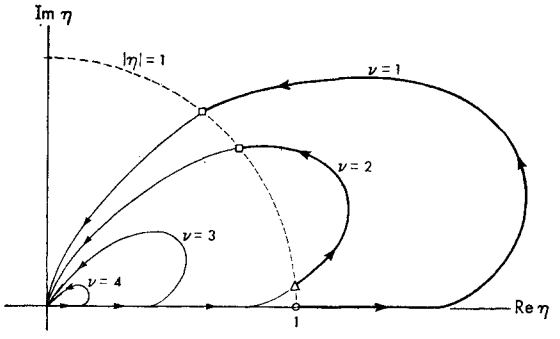

FIG. 5. A guess at the form of the trajectories of the first four p-wave eigenvalues $\eta_\nu(W)$ in the complex plane, for some typical attractive short-range potential. The arrows indicate direction of the trajectories as W increases from $-\infty$ to $\infty+i\epsilon$. Here there is both a $\nu=1$ bound state (small circle) and a $\nu=2$ sharp resonance (small triangle). The Born series diverges (darkened trajectories) until $\eta_1(W)$ and $\eta_2(W)$ both re-enter the unit circle (small squares).

$\delta_{(\nu)}(E)$:

$$\delta(E)=\sum_\nu \delta_{(\nu)}(E), \tag{97}$$

$$\delta_{(\nu)}(E)=-\arg[1-\eta_\nu(E+i\epsilon)]. \tag{98}$$

For $E\to\infty$ it was shown (part K, Sec. VI) that all the $\eta_\nu(E+i\epsilon)$ vanish and, hence,

$$\delta(\infty)=0. \tag{122}$$

For finite E we saw in part (J) that each $\eta_\nu(E+i\epsilon)$ stays in the upper (or lower)half plane, and for $E\to 0$ the $\eta_\nu(E+i\epsilon)$ approach positive (or negative) constants $\eta_\nu(0)$ [parts (E), (M)]. It follows then from (98) that

$$\delta_{(\nu)}(0)=\begin{cases}0 & \eta_\nu(0)<1\\ \pi & \eta_\nu(0)>1\end{cases}, \tag{123}$$

and so

$$\delta(0)=n\pi, \tag{124}$$

where n is the number of $\eta_\nu(0)$ greater than one. But the remarks after part (E) in Sec. VI made it clear that there must be precisely one bound state for each such $\eta_\nu(0)$, so that n is the total number of bound states.

The one special case not yet accounted for arises if some eigenvalue has

$$\eta_\nu(0)=1. \tag{125}$$

For S-wave scattering, Eq. (95) shows that as $E\to 0$

$$\eta_\nu(E+i\epsilon)\to 1+i\pi C_\nu E^{1/2}, \tag{126}$$

where $C_\nu>0$; hence, in this case

$$\delta_{(\nu)}(0)=\pi/2, \tag{127}$$

and we speak of an S-wave resonance at zero energy. On the other hand, for scattering with $l\neq 0$, Eq. (95) and part (E) show that as $E\to 0$

$$\eta_\nu(E+i\epsilon)\to 1+\eta_\nu'(0)E, \tag{128}$$

where $\eta_\nu'(0)>0$; hence, in this case

$$\delta_{(\nu)}(0)=\pi, \tag{129}$$

and we regard this as a zero-energy bound state.

So we see that (124) holds if we let n be the number of bound states, including zero-energy bound states for $l\neq 0$, and counting an S-wave resonance at zero energy as half a bound state. This is the full Levinson result.[19]

IX. CONDITIONS FOR CONVERGENCE

We would like to be able to use either experimental information or a knowledge of the interaction to decide *directly* when the Born series diverges, and when the divergence has been cured. Our condition (15) is all very well, but it does not accomplish this aim. Unfortunately, we do not know any complete solution to this problem, but we will present some partial answers here.

Suppose first that we measure the phase shift and all binding energies, and would then like to know whether the *original* Born series converges at a given W. For $W<0$ with a purely attractive interaction the answer is trivial; the Born series diverges from $W=0$ down to the lowest bound state, and converges below that. For scattering problems (where $W=E+i\epsilon$) there is no simple answer, unless we are willing to assume that the full-phase shift $\delta(E)$ arises mostly from the biggest elemental phase shift $\delta_1(E)$. In this case we can calculate the biggest eigenvalue $\eta_1(E)$ from (101) (assuming no bound state) and we get

$$|\eta_1(E+i\epsilon)|^2=1-2F(E)\cos\delta(E)+F^2(E),$$
$$F(E)=\exp\left\{-\frac{\mathrm{P}}{\pi}\int_0^\infty dE'\frac{\delta(E')}{E'-E}\right\}. \tag{130}$$

We must check whether this $|\eta_1|^2$ is less than one if we want to decide whether the original Born series converges; however, the series definitely diverges if $\cos\delta_{(1)}$ is negative, i.e., if $\delta\simeq\delta_{(1)}$ and if

$$\pi/2\leqq|\delta(E)|\leqq\pi. \tag{131}$$

These conclusions apply only to the Born series for $T(W)$; the presence of fictitious elementary particles invalidates them for $T_1(W)$.

Now suppose instead that we know the interaction V or (V_1), but do not have any experimental information. The complete problem of finding when the Born series converges is certainly at least as hard as that of locating all bound states. But it is possible to be sure for some values of $W<0$ that the Born series converges. (This was first realized by Bargmann,[20] and elaborated by Schwinger.[21]) For if we calculate $\tau_2(W)$ [see (80)] and

[19] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1947).
[20] V. Bargmann, Proc. Nat. Acad. Sci. **38**, 961 (1952).
[21] J. Schwinger, Proc. Nat. Acad. Sci. **47**, 122 (1961).

find that for some $W<0$,

$$\tau_2(W)<1, \tag{132}$$

then we can be sure from (79) that none of the $\eta_\nu^2(W)$ are greater than one; the Born series, thus, converges at W, and, hence, at all lower energies. The sum rule (79) also tells us that the number of bound states below W is less than $\tau_2(W)$. [This is a somewhat improved version of Schwinger's proof; Schwinger converts (14) into a self-adjoint eigenvalue problem by defining

$$K'(W)=|V|^{1/2}(W-H_0)^{-1}|V|^{1/2}$$

and using the fact that the Born series must converge for V if it does for $|V|$. Our trick of defining $K(W)$ by (78) avoids the troublesome business of defining $|V|$ for a general interaction.]

It should be emphasized that if $\tau_2(W)>1$ then the Born series may still converge, since a few eigenvalues smaller than one may still conspire to give a large τ_2. If we suspect that this is the case, we should calculate $\tau_4(W)$, $\tau_6(W)$, $\cdots$, since if all $|\eta_\nu|$ are less than one then this sequences must eventually reach a value less than one. But of course we do not know where to stop.

Unfortunately, there does not seem to be any comparable decision procedure available for scattering problems, where the eigenvalues $\eta_\nu(E+i\epsilon)$ are complex. Perhaps one exists, but I have not been able to discover it.

Another possible way of learning that the Born series converges is to compute the trace $\tau(W)$ [see (18)] and find that it is less than one. The discussion of Sec. III makes it clear that this is a sufficient but not a necessary criterion for convergence; in particular it is useless for scattering problems where $\tau(E+i\epsilon)$ is infinite. However, it leads to an amusing result for bound states. If $\tau(W)<1$ for some $W<0$, then the Born series converges at and below W, and hence, W must be less then the lowest bound-state energy $-B$. Using (A2) and (A3) (Appendix A), we obtain an upper bound on the binding energy in potential problems,

$$B<(m^3/8\pi^2)\left[\int d^3\mathbf{r}V^2(\mathbf{r})\right]^2, \tag{132}$$

which may be useful in conjunction with the lower bounds on B provided by variational calculations. Equation (132) may be used with any local potential $V(\mathbf{r})$ for which the integral exists, and holds for all bound states.

X. PRACTICAL CALCULATIONS

Our ultimate goal is to be able to perform relativistic strong-interaction calculations by perturbation theory. But it seems appropriate at this point to pause and show that the methods developed so far can be used in practical nonrelativistic calculations of binding energies and phase shifts.

Suppose we choose the bare vertices well enough so that the Born series (12) for $T_1(W)$ converges. Then it is reasonable as a first approximation to estimate

$$T_1(W)\cong V_1(W). \tag{133}$$

Inserting this in the exact formulas (8), (9), (10), we get

$$\begin{aligned}T(W)&\cong V_1(W)\\&\quad+N^{-2}(W)\Delta(W)V_1(W)|\Gamma(W)\rangle\langle\bar{\Gamma}(W)|V_1(W)\\&=V+\frac{J(W)}{1-J(W)}V|\Gamma(W)\rangle\langle\bar{\Gamma}(W)|V,\end{aligned} \tag{134}$$

and

$$\begin{aligned}J(W)&\cong N^{-2}(W)\langle\bar{\Gamma}(W)|V_1(W)\frac{1}{W-H_0}V_1(W)|\Gamma(W)\rangle\\&=\langle\bar{\Gamma}(W)|V\frac{1}{W-H_0}V|\Gamma(W)\rangle.\end{aligned} \tag{135}$$

[To get (134) and (135), we have used the identities

$$V_1(W)|\Gamma(W)\rangle=N(W)V|\Gamma(W)\rangle, \tag{136}$$

$$\langle\bar{\Gamma}(W)|V_1(W)=\langle\Gamma(W)|VN(W). \tag{137}$$

All factors $N(W)$ just cancel out, as promised in Sec. II.]

With (134) and (135), we can calculate the S matrix [see Eq. (4)] as

$$\begin{aligned}S_{n'n}(E)&\cong S_{n'n}{}^B(E)-2\pi i\frac{J(E+i\epsilon)}{1-J(E+i\epsilon)}\\&\quad\times\langle En'|V|\Gamma(E+i\epsilon)\rangle\langle\bar{\Gamma}(E+i\epsilon)|V|En\rangle;\end{aligned} \tag{138}$$

where S^B is the S matrix in the usual Born approximation,

$$S_{n'n}{}^B(E)=\delta_{n'n}-2\pi i\langle En'|V|En\rangle. \tag{139}$$

Also, if there is a bound state, its binding energy B can be calculated from (135) as the root of

$$J(-B)=1. \tag{140}$$

We will apply these results to the case of a spinless particle of mass m moving with orbital angular momentum l in a local central potential $V(r)$. In this case (135), (138), and (139) become

$$\begin{aligned}J(W)&=-2imk\int_0^\infty dr\int_0^\infty dr'\bar{\Gamma}(r,W)V(r)\\&\quad\times rj_l(kr_<)h_l^{(1)}(kr_>)r'V(r')\Gamma(r',W),\end{aligned} \tag{141}$$

$$\begin{aligned}e^{2i\delta(E)}=S(E)&=S^B(E)-4mki\frac{J(E+i\epsilon)}{1-J(E+i\epsilon)}\\&\quad\times\left\{\int_0^\infty dr\bar{\Gamma}(r,E+i\epsilon)V(r)rj_l(kr)\right\}\\&\quad\times\left\{\int_0^\infty drrj_l(kr)V(r)\Gamma(r,E+i\epsilon)\right\},\end{aligned} \tag{142}$$

$$S^B(E)=1-4mki\int_0^\infty r^2 j_l^2(kr)V(r)dr. \tag{143}$$

Here k is defined, in general, by (74); it should be taken real and positive in (142) and (143), and equal to $i(2mB)^{1/2}$ in (140). Also

$$\langle r|\Gamma(W)\rangle\equiv\Gamma(r,W), \tag{144}$$

$$\langle\bar{\Gamma}(W)|r\rangle\equiv\bar{\Gamma}(r,W). \tag{145}$$

It only remains to describe how we guess at a choice of $|\Gamma(W)\rangle$ and $\langle\bar{\Gamma}(W)|$. In general, we will always choose their normalization to agree with that of the "ideal" choice, i.e.,

$$\langle\bar{\Gamma}(W)|V|\Gamma(W)\rangle=1, \tag{53}$$

or in potential problems

$$\int_0^\infty \bar{\Gamma}(r,W)V(r)\Gamma(r,W)dr=1. \tag{146}$$

Our guess at the form of the bare vertices must also be conditioned by the requirement that they approximate the eigenvector Ψ_1. For example, in potential problems, we know that $\psi_1(r,W)$ must have the properties (72) and (73) so as a first crude guess we may try

$$\Gamma(r,W)=C(W)r^{l+1}e^{ikr} \tag{147}$$

$$\bar{\Gamma}(r,W)=\Gamma^*(r\,W^*)=C^*(W^*)r^{l+1}e^{ikr}. \tag{148}$$

[Equation (148) follows from the fact that $(2mW^*)^{1/2}$ must be chosen to have positive imaginary part, so that it equals $-([2mW]^{1/2})^*$.] The normalization condition (146) then gives

$$\frac{1}{C(W)C^*(W^*)}=\int_0^\infty r^{2l+2}e^{2ikr}V(r)dr, \tag{149}$$

so inserting (147)–(149) in (141) and (142), we get

$$J(W)\cong-2imk\int_0^\infty dr\int_0^\infty dr' r^{l+2}e^{ikr}V(r)j_l(kr_<)h_l^{(1)}(kr_>)r'^{l+2}e^{ikr'}V(r')\Big/\int_0^\infty r^{2l+2}e^{2ikr}V(r)dr, \tag{150}$$

$$e^{2i\delta(E)}=S(E)\cong S^B(E)-4imk\frac{J(E+i\epsilon)}{1-J(E+i\epsilon)}\left[\int_0^\infty dr e^{ikr}r^{l+2}V(r)j_l(kr)\right]^2\Big/\left[\int_0^\infty dr e^{2ikr}r^{2l+2}V(r)\right]. \tag{151}$$

With these formulas we expect to get a fair approximation to the lowest bound-state energy, and to the phase shift $\delta(E)$ for any energy for which not more than one eigenvalue $\eta_r(E+i\epsilon)$ is outside the unit circle. In particular, the scattering length a_S is defined by

$$S(E)\underset{E\to0}{\to}1-2ia_Sk^{2l+1}. \tag{152}$$

Comparing with (151), we get

$$a_S\cong\frac{a_{SB}}{1-J(0)}, \tag{153}$$

where a_{SB} is the Born approximation value

$$a_{SB}=\frac{2m}{(2l+1)!!^2}\int_0^\infty r^{2l+2}V(r)dr. \tag{154}$$

Equation (153) is expected to be a fair approximation as long as the potential is not so strong that it (or $-V(r)$) can give more than one bound state, whereas the presence of *any* bound states would certainly make a_{SB} a very bad approximation to the scattering length.

These formulas seem to do well on comparison with exact calculations. For S-wave scattering by a Yukawa potential,

$$V(r)=\frac{-\lambda}{2mar}e^{-r/a}, \tag{155}$$

we can calculate (150) easily, and get

$$J(W)=\frac{\lambda}{2(1-ika)}=\frac{\lambda}{2(1-i[2ma^2W]^{1/2})}. \tag{156}$$

The binding energy B of the deepest bound state is given by (140) as

$$\kappa a=\frac{\lambda}{2}-1;\quad \kappa=(2mB)^{1/2}. \tag{157}$$

The coupling strength required to give a bound state with zero energy is $\lambda_0=2$, as compared with the exact value[22] $\lambda_0=1.68$. The agreement looks even better in Fig. 6, where we plot κa versus λ for the exact answers[23] and for Eq. (157).

The Born approximation scattering length (154) in this case is

$$a_{SB}=-\lambda a. \tag{158}$$

Our corrected scattering length (153) then is

$$a_S=-\lambda a/(1-\lambda/2). \tag{159}$$

Both values are plotted in comparison with the exact values[24] in Fig. 7; our new Born approximation (159)

[22] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[23] R. G. Sachs and M. Goeppert-Mayer, Phys. Rev. **53**, 991 (1938).
[24] We have used results of Blatt and Jackson, reference 22, extrapolating slightly.

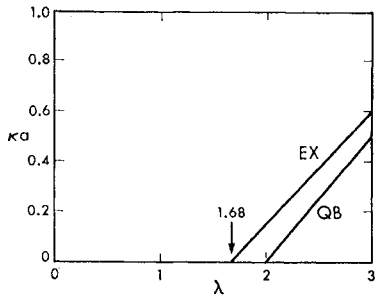

FIG. 6. The lowest S-wave bound state, for the Yukawa potential

$$V(r)=-\frac{\lambda}{2mar}e^{-r/a},$$

as a function of λ. (Here the binding energy is $\kappa^2/2m$.) We compare the exact answer (EX) with the result of using the first Born approximation (QB) after insertion of a quasiparticle with the bare vertices (147)–(149).

certainly does much better than the original Born approximation (158).

For infinite range a, the Yukawa potential goes over into the Coulomb potential (108), provided that we keep λ/a constant and equal to

$$\lambda/a=2mZe^2. \tag{160}$$

In this case (156) gives

$$J(-B)=\frac{2mZe^2a}{2(1+[2ma^2B]^{1/2})}\xrightarrow{a\to\infty}Ze^2(m/2B)^{1/2}. \tag{161}$$

The binding energy of the lowest bound state is then given by (140) as

$$B=Z^2e^4m/2, \tag{162}$$

which is exactly the right answer! This striking success in calculating the hydrogen atom binding energy in first-order perturbation theory is just due to the fact that our guess (147) and (148) happens to be the "ideal" choice for the Coulomb potential, since it equals the exact solution (109) for $\nu=l+1$.

It must be realized that these crude results are capable of infinite refinement. All that is needed is to keep successively higher terms in the series

$$T_1(W)=V_1(W)+V_1(W)[W-H_0]^{-1}V_1(W)+\cdots, \tag{12}$$

and substitute them in the exact expressions (8)–(10). This sequence of approximations will converge for a range of energy and coupling parameter which may be conditioned by the choice of bare vertices, but will certainly be much wider than the range of validity of the ordinary Born approximations. No matter how crude the choice of bare vertices [and our choice (147) and (148) is far from ideal for short-range forces] this scheme of successive approximations will give exactly correct answers where it converges at all. (Of course, each successive approximation depends on the choice of $|\Gamma\rangle$ and $\langle\Gamma|$; only the final answer does not.)

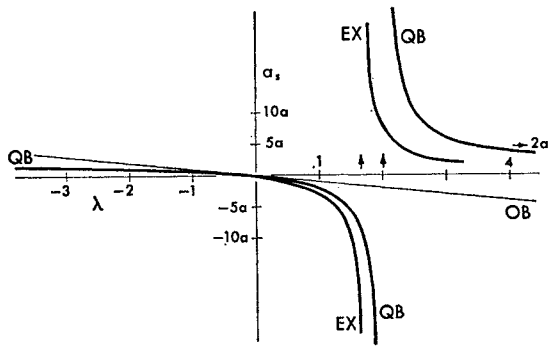

FIG. 7. The S-wave scattering length a_S, for the Yukawa potential

$$V(r)=-\frac{\lambda}{2mar}e^{-r/a},$$

as a function of λ. We compare the exact answer (EX), the ordinary Born approximation (OB), and the Born approximation (QB) used after the insertion of a quasiparticle with the bare vertices (147)–(149). The short vertical arrows at $\lambda=1.68$ and $\lambda=2$ mark the points where a_S becomes infinite because of an S-wave resonance at zero energy, for the cases (EX) and (QB), respectively.

In order to improve the range of validity and the speed of the convergence, we can try to make a better guess at our bare vertices. A particularly promising set for S-wave scattering by a potential of finite range a [see (75)] is

$$\Gamma(r,W)\sim\bar{\Gamma}(r,W)\sim e^{ikr}(1-e^{-r/a}). \tag{163}$$

This has a number of advantages:

(a) It has exactly the right asymptotic behavior [see (72) and (73)] both for $r\to 0$ and $r\to\infty$. The previous choice re^{ikr} made in (147) and (148) behaves correctly at $r\to 0$, but it is not quite right for $r\to\infty$, except of course for the Coulomb potential, where it is perfect.

(b) We remarked in Sec. V that any choice of $|\Gamma\rangle$ can be improved by applying the scattering kernel $[W-H_0]^{-1}V$ to it. If we try to improve re^{ikr} in this way, we get (163) for S-wave Yukawa scattering.

(c) Comparing with (119), we see that Eq. (163) gives the exact wave function $\psi_1(r,k)$ for the Hulthén potential. This means that the equation $J(-B)=1$ when evaluated to first order in $V_1(W)$ will yield the exact binding energy of the deepest state bound by the Hulthén potential, just as (147) and (148) gave the right answer for the Coulomb potential. More important, it means that (163) will not be very different in form from the ideal bare vertices for any reasonable potential of range a, including the Yukawa potential (155). Thus, it should give rapid convergence for all energies and coupling constants such that all but one eigenvalue $\eta_\nu(W)$ lies well within the unit circle.

A program of calculation of phase shifts and binding energies based on Eqs. (140)–(146) and (163) has been begun, and will be reported in a later paper.

XI. THE FREDHOLM CHOICE

We have seen in Sec. V that if the reduced interaction V_1 is formed by subtracting a *single* separable interaction $V_S=V|\Gamma\rangle\langle\bar{\Gamma}|V$ from V, and if V_S is subject to certain Hermiticity and analyticity conditions, then the most we can do to improve the convergence of the Born series is to choose V_S so that the greatest eigenvalue η_1 of the original kernel is reduced to zero. Now we shall show that if the Hermiticity condition on V_S is lifted then a *single* V_S can be chosen so that *any number* of eigenvalues are reduced to zero; this is essentially the Fredholm method.

Suppose we want the first p eigenvalues of the reduced kernel to be zero, and the others to be the original eigenvalues $\eta_\nu(W)$ with $\nu>p$. We will now show that the unique choice of $G_\nu(W)$ which accomplishes this is

$$G_\nu(W)=\prod_{\mu\neq\nu,\,\mu\leqq p}\left[1-\frac{\eta_\mu(W)}{\eta_\nu(W)}\right]^{-1} \qquad (\nu\leqq p) \quad (164)$$

$$G_\nu(W)=0 \qquad (\nu>p). \quad (165)$$

Furthermore, with such a choice of bare vertices, the propagator is

$$\Delta(W)=\prod_{\nu\leqq p}[1-\eta_\nu(W)]^{-1}. \quad (166)$$

Proof: We showed in Sec. IV that (165) is the necessary and sufficient condition for the $\eta_\nu(W)$ with $\nu>p$ to be still eigenvalues of the reduced kernel; if (165) holds then the other p reduced eigenvalues are the roots χ of a pth order polynomial in χ

$$Q(W,\chi)=\Delta(W,\chi^{-1})\prod_{\nu\leqq p}\left(1-\frac{\chi}{\eta_\nu(W)}\right)$$

$$=\left\{1+\sum_{\nu\leqq p}\frac{G_\nu(W)\eta_\nu(W)}{\chi-\eta_\nu(W)}\right\}\prod_{\nu\leqq p}\left(1-\frac{\chi}{\eta_\nu(W)}\right). \quad (167)$$

[See Eqs. (46) and (33).] These p roots will be zero if and only if $Q(\chi)$ is just proportional to χ^p. The constant of proportionality can be determined by noting from (167) that

$$\chi^{-p}Q(W,\chi)\xrightarrow[\chi\to\infty]{}\prod_{\nu\leqq p}\left(-\frac{1}{\eta_\nu(W)}\right),$$

so the necessary and sufficient condition for the first p reduced eigenvalues to be zero is

$$Q(W,\chi)=\prod_{\nu\leqq p}\left(-\frac{\chi}{\eta_\nu(W)}\right). \quad (168)$$

But (167) and (168) are equal if and only if they are equal at the p points $\eta_\nu(W)(\nu\leqq p)$. This gives the p equations

$$-G_\nu(W)\prod_{\mu\leqq p,\,\mu\neq\nu}\left[1-\frac{\eta_\nu(W)}{\eta_\mu(W)}\right]=\prod_{\mu\leqq p}\left(-\frac{\eta_\nu(W)}{\eta_\mu(W)}\right),$$

whose solution is (164). The propagator $\Delta(W)$ is easily derived from (167), since

$$\Delta(W)=\Delta(W,1)=Q(W,1)\prod_{\nu\leqq p}\left(1-\frac{1}{\eta_\nu(W)}\right)^{-1}. \quad (169)$$

Using (168) then gives (166).

It is never possible to get the coefficients (164) and (165) with an Hermitian reduced interaction $V_1(W)$, since then for $W<0$ all of the $G_\nu(W)/\eta_\nu(W)$ would have to be of the same sign. But (167) and (168) show that for $p>1$,

$$\frac{\partial}{\partial\chi}\bigg|_{\chi=0}Q(W,\chi)=-\sum_{\nu\leqq p}G_\nu(W)/\eta_\nu(W)=0, \quad (170)$$

which would then be impossible.

When V is a short-range potential, the $\eta_\nu(W)$ vanish for $\nu\to\infty$ rapidly enough so that the products (164) and (166) converge in the limit $p\to\infty$. [See Eq. (96).] Then we see that there is a unique choice of the coefficients G_ν:

$$G_\nu(W)=\prod_{\nu\neq\mu}\left[1-\frac{\eta_\mu(W)}{\eta_\nu(W)}\right], \quad (171)$$

such that *all eigenvalues of the reduced kernel are zero.* The reduced kernel $V_1[W-H_0]^{-1}$ must then have Fredholm determinant unity, since all its powers have trace zero. It follows then from (13) that the original Fredholm determinant is just

$$D(W)=\Delta^{-1}(W)=\sum_\nu[1-\eta_\nu(W)]. \quad (172)$$

Our argument is nonrigorous, but the answer is certainly correct.

It should be mentioned that if the biggest eigenvalue $\eta_1(W)$ is very much larger than the next biggest $\eta_2(W)$, then (171) gives

$$\begin{aligned}&G_1(W)=1+O[\eta_2(W)/\eta_1(W)],\\ &G_\nu(W)=O[\eta_2(W)/\eta_1(W)] \quad \nu\neq1.\end{aligned} \quad (173)$$

Hence, in this case the Fredholm choice gives essentially the same coefficients $G_\nu(W)$ as did the "ideal" choice of Sec. V.

For a long time it has been implicitly known that the Fredholm method can be understood as a special case of the quasiparticle method, at least for short-range potentials. The scattering kernel in position space in the lth partial wave is

$$\langle r|G(W)V|r'\rangle=-ikrr'j_l(kr_<)h_l^{(1)}(kr_>)V(r'), \quad (174)$$

where

$$G(W)=(W-H_0)^{-1}. \quad (175)$$

We may define a reduced kernel as $G_1(W)V$, where $G_1(W)$ is

$$G_1(W)=G(W)-G_S(W), \quad (176)$$

$G_S(W)$ being the separable operator defined by[25]

$$\langle r|G_S(W)V|r'\rangle=-ikrr'j_l(kr)h_l^{(1)}(kr')V(r'). \quad (177)$$

It is easy to see then that the reduced kernel must have Fredholm determinant unity, since its matrix elements are

$$\langle r|G_1(W)V|r'\rangle=-ikrr'\theta(r-r')j_l(kr')h_l^{(1)}(kr)V(r'),$$

and obviously all its powers have zero trace. By the same algebra that gave (42), we can show here that the propagator of the quasiparticle corresponding to (176) is

$$\Delta(W)=1-ik\int_0^\infty dr\int_0^\infty dr'r'h_l^{(1)}(kr')T(r',r)rj_l(kr), \quad (179)$$

where as usual

$$T=V+VGT.$$

This is identical to one of the formulas[26] for the inverse of the Jost function $f(-k)$. But the Fredholm determinant $D(W)$ of the original kernel is given by (13) as $\Delta^{-1}(W)$, since $D_1(W)=1$. This proves the well-known result[25] that

$$D(W)=f(-k). \quad (180)$$

It is not at all clear to me how the choice (177) corresponds to the formula (171) for $G_r(W)$, nor how in a more general problem we can make such a felicitous guess.

Which choice of bare vertices should we aim at approximating, the "ideal" choice of Sec. V or the Fredholm choice described in this section? It is not at all clear which choice gives the most rapidly convergent reduced Born series (12), but there is a different sort of reason for preferring the "ideal" choice. We showed in part (b) of Sec. V that we can always make any trial bare vertex $\Gamma^{(0)}$ more and more "ideal" by repeated multiplication with the scattering kernel $[W-H_0]^{-1}V$. There is no equally general procedure for approximating the Fredholm coefficients (171), though we have seen that there is a trick available in potential theory which makes the reduced kernel of the Volterra type, and which must therefore yield (171). Until we learn how to use this trick more generally the choice of Sec. V is to be preferred; it makes only one reduced eigenvalue zero, but for just that reason it is easy to approximate. And let us stress once again that even a very rough guess at a more or less ideal $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$ will enable us to get *exact* results by using more and more terms of the reduced Born series. We make no claim that this is the best way of solving two-body problems, but we hope that it may be a way (and perhaps the only way) of solving harder problems.

[25] For S waves, see R. Jost and A. Pais, reference 5. For general l, see R. Newton, reference 10.

[26] See reference 10, Eqs. (4.4) and (4.8).

APPENDIX A: CALCULATION OF $\tau(W)$

We consider first the three-dimensional problem of a particle of mass m being scattered by a local potential $V(\mathbf{r})$. In this case the trace $\tau(W)$ is given by (18) as

$$\tau(W)=4m^2\int d^3\mathbf{q}\langle\mathbf{q}|V^2|\mathbf{q}\rangle/|k^2-\mathbf{q}^2|^2, \quad (A1)$$

where $k^2=2mW$. In position space this becomes

$$\tau(W)=I(W)\int d^3\mathbf{r}V^2(\mathbf{r}), \quad (A2)$$

here

$$I(W)=[4m^2/(2\pi)^3]\int d^3\mathbf{q}(1/|k^2-\mathbf{q}^2|^2)$$
$$=m^2/2\pi|\mathrm{Im}k|. \quad (A3)$$

So $\tau(W)$ is finite if (19) holds, and if W is off the positive real axis.

To go over into the partial-wave problem for a central potential $V(r)$, we may note that $|W-H_0|^{-2}$ becomes in position space

$$\langle\mathbf{r}'||W-H_0|^{-2}|\mathbf{r}\rangle=\frac{4m^2}{(2\pi)^3}\int d^3\mathbf{q}\frac{\exp[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')]}{|k^2-\mathbf{q}^2|^2}$$
$$=\frac{m^2}{\pi\,\mathrm{Im}k^2}\mathrm{Im}\left(\frac{e^{ik\rho}}{\rho}\right), \quad (A4)$$

where k is the square root of $2mW$ with positive imaginary part, and $\rho=|\mathbf{r}-\mathbf{r}'|$. Using the well-known partial-wave expansion of $e^{ik\rho}/ik\rho$, we find that

$$\langle\mathbf{r}'||W-H_0|^{-2}|\mathbf{r}\rangle=\sum_{l=0}^\infty[(2l+1)/4\pi]$$
$$\times P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')I_l(r,r';k); \quad (A5)$$

$$I_l(r,r';k)=(4m^2/\mathrm{Im}k^2)$$
$$\times\mathrm{Im}[ikj_l(kr_<)h_l^{(1)}(kr_>)]. \quad (A6)$$

Hence, the trace $\tau(W)$ is given by a sum of partial-wave traces

$$\tau(W)=\sum_{l=0}^\infty(2l+1)\tau(W,l), \quad (A7)$$

where

$$\tau(W,l)=\int_0^\infty V^2(r)I_l(r,r;k)r^2dr. \quad (A8)$$

It is easily seen that $I_l(r,r;k)$ approaches a constant as $r\to 0$, but behaves like r^{-2} as $r\to\infty$, so that $\tau_l(W)$ exists if (20) holds and if W is not real and positive. For $W<0$, $k=i\kappa$, where $\kappa=(2m|W|)^{1/2}>0$. In this case, the partial-wave trace stays finite, because the numerator as well as the denominator of (A6) vanish. For example,

the S-wave trace is

$$\tau(W,0)=\frac{2m^2}{\mathrm{Im}k^2}\int_0^\infty V^2(r)\,\mathrm{Im}\left(\frac{e^{2ikr}-1}{ik}\right)dr, \quad \text{(A9)}$$

and when $k\to i\kappa$ this becomes

$$\tau(W,0)=(m^2/\kappa^3)\int_0^\infty V^2(r)[1-e^{-2\kappa r}(1+2\kappa r)]dr. \quad \text{(A10)}$$

This is finite, even for the Coulomb potential, for which we get

$$\tau(W,0)=2Z^2e^4m^2/\kappa^2. \quad \text{(A11)}$$

In this particular case the actual radius of convergence of the Born series [given by $|\eta_1|^{-1}$; see Eq. (110)] is greater than $\tau^{-1/2}$ by a factor of $\sqrt{2}$, in accordance with the general remarks of Sec. III.

APPENDIX B: CALCULATION OF $\tau_2(W)$

It is easy to see that the operator $K(W)$, defined by (78), is an L^2 kernel whenever $[W-H_0]^{-1}V$ is, because the trace $\tau_2(W)$, defined by

$$\tau_2(W)=\mathrm{Tr}\{K(W)K^\dagger(W)\}=\sum_{nn'}\int_0^\infty dE\int_0^\infty dE' \times|\langle En|V|E'n'\rangle|^2/|E-W||E'-W|, \quad \text{(B1)}$$

may be written as

$$\tau_2(W)=\tau(W)-\tfrac{1}{2}\sum_{nn'}\int_0^\infty dE\int_0^\infty dE'|\langle En|V|E'n'\rangle|^2 \times[|W-E|^{-1}-|W-E'|^{-1}]^2, \quad \text{(B2)}$$

so that

$$\tau_2(W)<\tau(W). \quad \text{(B3)}$$

For example, for the Yukawa potential (155), (B1) gives

$$\tau_2(W)=(\lambda^2/4\pi^4)\int d^3\mathbf{q}\int d^3\mathbf{q}' \times 1/|\mathbf{q}^2-2mW||\mathbf{q}'^2-2mW|[1+a^2(\mathbf{q}-\mathbf{q}')^2]^2.$$

With $W<0$, the integral can be done in closed form; we get

$$\tau_2(W)=\lambda^2/2(1+2\kappa a);\quad \kappa=(-2mW)^{1/2}. \quad \text{(B4)}$$

On the other hand, $\tau(W)$ is given here by (A2) and (A3) as

$$\tau(W)=\lambda^2/4\kappa a, \quad \text{(B5)}$$

which verifies (B3). The fact that $\tau(W)$ becomes infinite as $\kappa\to 0$ does not imply that the Born series breaks down there, because Sec. VI shows that $\eta_\nu(0)$ is finite; in fact, we even see from (B4) that $\tau_2(0)$ is finite.

Note added in proof. We have observed in Sec. III that the scattering kernel K is not L^2 for the full three-dimensional Coulomb-scattering problem. However, it can be shown that the kernel here is completely continuous [in the sense of functional analysis] because $(K^\dagger K)^2$ has a finite trace, and that, therefore, all the important theorems of this paper hold even in this case. However, for Coulomb scattering there are serious mathematical difficulties associated with the limit $\epsilon\to 0$ in Eq. (4).